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THE BORSUK-ULAM THEOREM FOR A Z_q-MAP FROM A Z_q-SPACE TO S^{2n+1}

TEIICHI KOBAYASHI[1]

ABSTRACT. J. W. Walker obtained in [2] a generalization of the Borsuk-Ulam theorem. The purpose of this note is to prove a mod q version of Walker's theorem.

1. Introduction. Let S^{2n+1} be the $(2n+1)$-dimensional standard sphere in complex $(n+1)$-space C^{n+1}. For an integer $q > 1$, let $T\colon S^{2n+1} \to S^{2n+1}$ be the transformation defined by

$$T(z_0, z_1, \ldots, z_n) = \left(e^{2\pi i/q}z_0, e^{2\pi i/q}z_1, \ldots, e^{2\pi i/q}z_n\right),$$

where $z_0, z_1, \ldots, z_n$ are complex numbers with $\sum_{i=0}^n |z_i|^2 = 1$. T acts freely on S^{2n+1} and generates a cyclic group Z_q of order q.

Let X be any Z_q-space. Let $\theta = 1 + T + \cdots + T^{q-1}$ and $\Delta = T - 1$ be elements in the group ring of Z_q. Then it is easily seen that θ and Δ determine chain maps $\theta_\# = 1_\# + T_\# + \cdots + T_\#^{q-1}$ and $\Delta_\# = T_\# - 1_\#$ on the singular chain complex of X, which we denote simply by θ and Δ. Hence the homology homomorphisms θ_* and Δ_* are well defined. They are natural with respect to Z_q-maps. Henceforth we use singular chain groups $S_i(X; Z_q)$ and singular homology groups $H_i(X; Z_q) = Z_i(X; Z_q)/B_i(X; Z_q)$ with coefficients in Z_q. The chain maps above satisfy

$$\theta\theta = \theta\Delta = \Delta\theta = 0. \tag{1.1}$$

THEOREM 1. *Let q be an integer > 1 and Z_q be the cyclic group of order q generated by T. Define $\theta = 1 + T + \cdots + T^{q-1}$. Let $X\ (\neq \varnothing)$ be a Z_q-space and $f\colon X \to S^{2n+1}$ be a Z_q-map. Assume, for any integer i with $0 < i < 2n+1$ and any element $\alpha \in H_i(X; Z_q)$, $\theta_*(\alpha) = 0$ implies $\alpha = 0$. Then there exists an element $\beta \in H_{2n+1}(X; Z_q)$ such that $f_*(\beta) \neq 0$ and $\theta_*(\beta) = 0$.*

COROLLARY 2 (Mod q BORSUK-ULAM THEOREM). *If $f\colon S^{2m+1} \to S^{2n+1}$ is a Z_q-map, then $m \leqslant n$.*

PROOF OF COROLLARY 2. Suppose $m > n$. Since $H_i(S^{2m+1}; Z_q) = 0$ for any integer i with $0 < i < 2m+1$, the assumption of Theorem 1 is clearly satisfied for $X = S^{2m+1}$. Hence it follows that there exists an element $\beta \in H_{2n+1}(S^{2m+1}; Z_q)$ such that $f_*(\beta) \neq 0$. But this is impossible, since $H_{2n+1}(S^{2m+1}; Z_q) = 0$.

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[1] This paper is dedicated to Professor Itiro Tamura on his 60th birthday.

2. Regular cell decomposition of S^{2n+1}. Let q be an integer > 1 and Z_q be the cyclic group of order q generated by T. In this section, we recall the Z_q-equivariant regular cell decomposition of S^{2n+1} (cf. [**1**, Chapter 5, §5]). First, we construct a cell decomposition for S^1. Choose $e_0, Te_0, \ldots, T^{q-1}e_0$, which are 0-cells of S^1. Then we get in the obvious way e_1, $Te_1, \ldots, T^{q-1}e_1$, which are 1-cells of S^1, such that $\partial T^j e_1 = T^{j+1}e_0 - T^j e_0 = T^j \Delta e_0$.

Suppose by induction that the $(2k-1)$-skeleton of S^{2k+1} has the regular cell decomposition. Since $S^{2k+1} = S^{2k-1} * S^1$ (where $*$ means join), we define the $2k$-cells of S^{2k+1} to be of the form $S^{2k-1} * T^j e_0 = T^j e_{2k}$ and the $(2k+1)$-cells of S^{2k+1} to be of the form $S^{2k-1} * T^j e_1 = T^j e_{2k+1}$, where $0 \leqslant j < q$. There are q cells in each dimension ranging from 0 to $2n+1$. We choose the orientation correctly such that

$$\partial T^j e_{2k} = \theta e_{2k-1}, \qquad \partial T^j e_{2k+1} = T^j \Delta e_{2k}, \tag{2.1}$$

where $0 \leqslant j < q$ and $0 \leqslant k \leqslant n$. The cell complex is Z_q-equivariant and regular. We use the same symbol $T^j e_k$ for the singular k-simplex $T^j \delta_k$, where $\delta_k\colon \Delta_k \to e_k$ is the standard homeomorphism from the standard k-simplex Δ_k to e_k.

3. Proof of Theorem 1. Let X $(\neq \varnothing)$ be a Z_q-space and $f\colon X \to S^{2n+1}$ be a Z_q-map. Assume, for any integer i with $0 < i < 2n+1$ and any element $\alpha \in H_i(X; Z_q)$, $\theta_*(\alpha) = 0$ implies $\alpha = 0$.

First, we will construct singular i-chains $s_i \in S_i(X; Z_q)$ for $i = 0, 1, \ldots, 2n+1$, such that $s_0, Ts_0, \ldots, T^{q-1}s_0$ are elementary 0-chains and

$$\partial T^j s_{2k} = \theta s_{2k-1}, \qquad \partial T^j s_{2k+1} = T^j \Delta s_{2k}, \tag{3.1}$$

for $0 \leqslant j < q$ and $0 \leqslant k \leqslant n$.

Pick a point $s_0 \in X$ and let maps $\Delta_0 \to T^j s_0$ $(j = 0, 1, \ldots, q-1)$ be singular elementary 0-chains, which are also denoted by $T^j s_0$. Since $\Delta T^j s_0$ is a singular cycle and since $\theta \Delta T^j s_0 = 0$ by (1.1), $\theta_*[\Delta T^j s_0] = 0$, where $[c]$ denotes the homology class of a singular cycle c. Hence we have $[\Delta T^j s_0] = 0$ by our assumption. So there exists a singular 1-chain $T^j s_1$ such that $\partial T^j s_1 = \Delta T^j s_0 = T^j \Delta s_0$. Now θs_1 is a singular cycle. In fact, $\partial \theta s_1 = \theta \partial s_1 = \theta \Delta s_0 = 0$ by (1.1). Since $\theta\theta s_1 = 0$ by (1.1), $\theta_*[\theta s_1] = 0$ and so $[\theta s_1] = 0$ by our assumption. Thus there exists a singular 2-chain $T^j s_2$ such that $\partial T^j s_2 = \theta s_1$. Inductively, we have singular i-chains $s_i \in S_i(X; Z_q)$ for $i = 0, 1, \ldots, 2n+1$, which satisfy (3.1).

Next, we will construct singular i-chains $d_i \in S_i(S^{2n+1}; Z_q)$ for $i = 0, 1, \ldots, 2n+1$, such that

$$\begin{gathered} T^j e_{2k} - f_\# T^j s_{2k} - \theta d_{2k} \in Z_{2k}\left(S^{2n+1}; Z_q\right), \\ T^j e_{2k+1} - f_\# T^j s_{2k+1} - \Delta T^j d_{2k+1} \in Z_{2k+1}\left(S^{2n+1}; Z_q\right), \end{gathered} \tag{3.2}$$

for $0 \leqslant j < q$ and $0 \leqslant k \leqslant n$, where $f_\#\colon S_i(X; Z_q) \to S_i(S^{2n+1}; Z_q)$ is a homomorphism of singular chain groups induced by f.

Since $T^j e_0 - f_\# T^j s_0$ is a singular cycle, we define $d_0 = 0$ and so $T^j e_0 - f_\# T^j s_0 - \theta d_0 \in Z_0(S^{2n+1}; Z_q)$. Clearly the sum of coefficients of this cycle is 0, and so it is a boundary. Hence there exists a singular 1-chain $T^j d_1 \in S_1(S^{2n+1}; Z_q)$ such that

$\partial T^j d_1 = T^j e_0 - f_\# T^j s_0 - \theta d_0$. Apply the chain map Δ to the equality. Then we obtain, by (1.1) and (2.1),

$$\partial \Delta T^j d_1 = T^j \Delta e_0 - f_\# T^j \Delta s_0 = \partial T^j e_1 - \partial f_\# T^j s_1,$$

and hence $\partial(T^j e_1 - f_\# T^j s_1 - \Delta T^j d_1) = 0$. Since $Z_1(S^{2n+1}; Z_q) = B_1(S^{2n+1}; Z_q)$, there exists a singular 2-chain $T^j d_2 \in S_2(S^{2n+1}; Z_q)$ such that $\partial T^j d_2 = T^j e_1 - f_\# T^j s_1 - \Delta T^j d_1$. Apply the chain map θ to the equality. Then we obtain, by (1.1) and (2.1),

$$\partial \theta T^j d_2 = \theta T^j e_1 - f_\# \theta T^j s_1 = \theta e_1 - f_\# \theta s_1 = \partial T^j e_2 - \partial f_\# T^j s_2.$$

Therefore $T^j e_2 - f_\# T^j s_2 - \theta T^j d_2 \in Z_2(S^{2n+1}; Z_q)$. Inductively, we obtain singular i-chains $d_i \in S_i(S^{2n+1}; Z_q)$ for $0 \leqslant i \leqslant 2n+1$ with the desired properties (3.2), because $H_i(S^{2n+1}; Z_q) = 0$ for $0 < i < 2n+1$.

Since the group $H_{2n+1}(S^{2n+1}; Z_q)$ is isomorphic to Z_q and is generated by $[\theta e_{2n+1}]$, $[e_{2n+1} - f_\# s_{2n+1} - \Delta d_{2n+1}] = x[\theta e_{2n+1}]$ for some integer x with $0 \leqslant x < q$. Apply θ_* to the equality. Then we have, by (1.1),

$$\left[\theta e_{2n+1} - f_\# \theta s_{2n+1}\right] = 0.$$

Since θe_{2n+1} and θs_{2n+1} are singular cycles, we have $f_*[\theta s_{2n+1}] = [\theta e_{2n+1}]$, which is a generator of $H_{2n+1}(S^{2n+1}; Z_q) \cong Z_q$ and is nonzero. If we set $\beta = [\theta s_{2n+1}]$, we have $f_*(\beta) \neq 0$ and $\theta_*(\beta) = 0$.

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Department of Mathematics, Faculty of Science, Kochi University, Kochi 780, Japan