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EXISTENCE OF A VOLUME PRESERVING DIFFEOMORPHISMS WITHOUT PERIODIC POINTS ON THREE-DIMENSIONAL MANIFOLDS

NOBUYA WATANABE

ABSTRACT. In this paper we show that on a smooth oriented closed 3-manifold there exists a volume preserving diffeomorphism without periodic points.

Introduction. J. F. Plante [3] showed that on a smooth oriented closed manifold M of dimension $\neq 3$, there exists a volume preserving diffeomorphism without periodic points if and only if the Euler characteristic $\chi(M) = 0$. In this note, we show that it is also valid in the three-dimensional case. That is, we prove the following

THEOREM. *Let M be a smooth oriented closed 3-manifold. Then M admits a volume preserving diffeomorphism without periodic points.*

The idea of the proof of the Theorem is mostly similar to Plante's. If $\chi(M) = 0$, by a theorem of Gromov [2, 3] there exists a nowhere vanishing divergence free vector field on M. This vector field can be perturbed to a nowhere vanishing divergence free vector field whose flow has at most a countable number of periodic orbits. Then, the set P of all rational multiples of periods of periodic orbits will be a countable set. If ϕ_t denotes the flow of the vector field and τ is a real number not contained in P then the diffeomorphism $f = \phi_\tau$ has no periodic points. Plante applied the Kupka-Smale theorem for divergence free vector fields due to Robinson [4] in order to perturb the vector field. Unfortunately the Kupka-Smale theorem for divergence free vector fields is valid only for dimension $\geqslant 4$. However, the countability of the number of periodic orbits is assured by a certain transversality of periodic orbits. Therefore, we will prove the Proposition (see §2) which states that a divergence free vector field is approximated by the divergence free vector fields with this transversality. Then we will apply the Proposition for the approximation.

1. Definition and notation. Let M be a smooth oriented closed manifold. We fix a volume form Ω on M. We set

$$\Psi^r(M) = \{C^r \text{ vector fields on } M\},$$
$$\Psi^r_\Omega(M) = \{X \in \Psi^r(M)\colon L_X\Omega = 0\},$$

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where L_X is the Lie derivative with respect to X. $\Psi^r(M)$ is endowed with C^r-topology and $\Psi^r_\Omega(M)$ with the induced topology. Then $\Psi^r(M)$ and $\Psi^r_\Omega(M)$ are both Baire space.

Let $X \in \Psi^r(M)$ and let ϕ_t be a flow of X.

DEFINITION. For a natural number N, a periodic orbit γ of ϕ_t is called *N-elementary* if no eigenvalue of the derivative of the nth iterate of the Poincaré map at $m \in \gamma$ is 1 for $1 \leqslant n \leqslant N$. A periodic orbit γ is called *elementary* if γ is N-elementary for any N.

We set

$$E\Psi^r_\Omega(M) = \{ X \in \Psi^r_\Omega(M);\ \text{all stationary points are hyperbolic and all periodic orbits are elementary} \}.$$

2. Proof.

PROPOSITION. *Let M be a smooth oriented closed 3-manifold. Then $E\Psi^r_\Omega(M)$ is a residual subset of $\Psi^r_\Omega(M)$.*

PROOF. The proof is completely parallel to those of Theorem 1 A(i) and B(ii) of Robinson [**4**]. First, the set of vector fields with hyperbolic zeros is dense and open in $\Psi^r_\Omega(M)$. The proof of Robinson's theorem clearly shows that the following lemma holds.

LEMMA. *Let $X \in \Psi^r_\Omega(M)$ and suppose that γ is a closed orbit of X. Let Σ be a transverse section to γ at a point $m \in \gamma$ and let k and N be natural numbers such that $N \geqslant k$. Then there exist neighborhoods O of X in $\Psi^r_\Omega(M)$ and V of m in M such that $R(k, N) = \{Y \in O$: all closed orbits of Y corresponding to points of period $\leqslant k$ of the Poincaré map for Y on $\Sigma \cap \overline{V}$ are N-elementary$\}$ is dense and open in O.*

For $\lambda > 0$, we set

$$E^r(\lambda) = \{ X \in \Psi^r_\Omega(M);\ \text{all stationary points are hyperbolic and all periodic orbits with period } \leqslant \lambda \text{ are elementary} \}.$$

By the Lemma, $R(k) = \bigcap_{N=k}^{\infty} R(k, N)$ is residual in O. Then, the same argument as in the proof of Robinson's theorem shows that $E^r(\lambda)$ is residual in $\Psi^r_\Omega(M)$. $E\Psi^r_\Omega(M) = \bigcap_{n=1}^{\infty} E^r(n)$ is residual in $\Psi^r_\Omega(M)$.

PROOF OF THE THEOREM. Since $\Psi^r_\Omega(M)$ is a Baire space, $E\Psi^r_\Omega(M)$ is dense in $\Psi^r_\Omega(M)$ by the Proposition. If $X \in E\Psi^r_\Omega(M)$, then the number of periodic orbits of X is at most countable by the finiteness theorem [**1**, p. 67]. Therefore, by the argument in the Introduction the proof is completed.

REMARK. All diffeomorphisms constructed here are homotopic to the identity. So, the following question is naturally raised.

Question. Which homotopy class of homeomorphisms contains a homeomorphism without periodic points?

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Department of Mathematics, School of Education, Waseda University, 1-6-1 Nishi-Waseda, Shinjuku-Ku, Tokyo 160, Japan