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ENGULFING AND FINITELY GENERATED GROUPS

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ABSTRACT. Let M be a simply connected 3-manifold and K a piecewise-linear, simple loop in the interior of M. It is shown that there is a piecewise-linear, homotopy 3-ball $\mathcal{B} \subset \overset{\circ}{M}$, such that $K \subset \overset{\circ}{\mathcal{B}}$ if and only if $\pi_1(M \backslash K)$ is finitely generated.

Let M be a 3-manifold and $K \subset \overset{\circ}{M}$. Say that K may be *engulfed* if there is a piecewise-linear, homotopy 3-ball $\mathcal{B} \subset \overset{\circ}{M}$ such that $K \subset \overset{\circ}{\mathcal{B}}$. The concept of engulfing is used in [**6** and **2**] to distinguish certain open, contractible 3-manifolds from $\mathbf{R}^3$. More specifically, each of the 3-manifolds contains a piecewise-linear, simple loop K which cannot be engulfed. This simple loop K also has the strange property that the fundamental group of the complement of K is not finitely generated. Below it is proved that this is not a coincidence. The proof is generalized from [**5**].

Interesting examples of M satisfying the hypotheses of the next theorem are contractible 3-manifolds, connected sums of contractible 3-manifolds, and the complements of Cantor sets in S^3 which have simply connected complements.

All manifolds and maps are assumed to be piecewise-linear.

THEOREM. *Let M be a simply connected, 3-manifold and K a piecewise-linear, simple loop in $\overset{\circ}{M}$. Then K may be engulfed if and only if $\pi_1(M \backslash K)$ is finitely generated.*

PROOF. First suppose that K may be engulfed by $\mathcal{B}$. Since K is piecewise-linear, clearly $\pi_1(\mathcal{B} \backslash K)$ is finitely generated. By Van Kampen's Theorem, $\pi_1(M \backslash K) \approx \pi_1(\mathcal{B} \backslash K)$, and one direction is proved.

Now suppose $\pi_1(M \backslash K)$ is finitely generated. By [**3**] there is a compact, piecewise-linear 3-manifold $Q \subset M \backslash K$, such that $\pi_1(Q) \to \pi_1(M \backslash K)$ is an isomorphism. Since M is orientable, Q too is orientable. Consider ∂Q. Since $\pi_1(Q) \neq 1$, Van Kampen's Theorem implies that ∂Q has at least one non-2-sphere component. Conversely, because Q is compact and orientable,

$$\operatorname{rank}(\operatorname{Image}(H_1(\partial Q) \to H_1(Q))) = \tfrac{1}{2}\operatorname{rank}(H_1(\partial Q)).$$

Since $H_1(Q) \approx H_1(M \backslash K) \approx \mathbf{Z}$, $Q = T \coprod (\coprod S^2)$, where $T \approx S^1 \times S^1$.

Suppose $\pi_1(T) \to \pi_1(Q)$ is a monomorphism. Let $N(K)$ be a regular neighborhood of K in $M \backslash Q$. It will be shown that T is parallel to $\partial N(K)$. Each component of ∂Q separates M. Hence, each component of $M \backslash \overset{\circ}{N}(K) \backslash \overset{\circ}{Q}$ intersects Q in exactly one component of ∂Q. Let P be the component of $M \backslash \overset{\circ}{N}(K) \backslash \overset{\circ}{Q}$ containing $\partial N(K)$.

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Clearly, P is not simply connected. Since $\pi_1(Q) \to \pi_1(M \backslash \mathring{N}(K))$ is an isomorphism, Van Kampen's Theorem implies that $P \cap Q = T$. And since $\pi_1(T) \to \pi_1(Q)$ is a monomorphism, $\pi_1(T) \to \pi_1(P)$ is a monomorphism. Again by Van Kampen's Theorem $\pi_1(Q) \approx \pi_1(M\backslash K) \approx \pi_1(Q) *_{\pi_1(T)} \pi_1(P)$ and $\pi_1(T) \to \pi_1(P)$ is an isomorphism. By the lemma below, $\pi_1(\partial N(K)) \to \pi_1(P)$ is also an isomorphism.

Let μ, λ be the meridional and longitudinal curves on $\partial N(K)$; suppose μ and λ intersect transversely at one point. Similarly, let μ', λ' be corresponding curves on T which are freely homotopic to μ, λ respectively. By [**4**] there are properly embedded annuli $A_1, A_2 \subset P$ with $\partial A_1 = \mu \cup \mu'$ and $\partial A_2 = \lambda \cup \lambda'$. By cut and paste techniques suppose $A_1 \cap A_2$ is a single arc. Let $N(A_1 \cup A_2)$ be a regular neighborhood of $A_1 \cup A_2$ in P. Clearly, $\mathcal{B}' = Q \cup N(A_1 \cup A_2) \cup N(K)$ has boundary a disjoint union of S^2's and $\pi_1(\mathcal{B}') = 1$. By drilling out holes in $\mathcal{B}'$ one may reduce the number of boundary components and produce the desired $\mathcal{B}$.

Finally, if $\pi_1(T) \to \pi_1(Q)$ is not a monomorphism, then an application of the Loop Theorem shows that

$$\mathbf{Z} \approx \pi_1(Q) \approx \pi_1(M \backslash K).$$

By Dehn's Lemma K bounds a disk in M, and a regular neighborhood of this disk is the desired $\mathcal{B}$. □

That $\mathcal{B}$ may be chosen to be a (real) 3-ball is equivalent to the Poincaré Conjecture (see [**1**]).

LEMMA. *Let P be a connected 3-manifold with boundary (not necessarily compact). Let F and G be distinct components of ∂P such that $F \approx G \approx S^1 \times S^1$. If $\pi_1(F) \to \pi_1(P)$ is an isomorphism, then $\pi_1(G) \to \pi_1(P)$ is an isomorphism.*

PROOF. First suppose $\pi_1(G) \to \pi_1(P)$ is not a monomorphism. By the Loop Theorem there is an essential simple loop $C \subset G$ which bounds a disk D in P. Because $G \approx S^1 \times S^1$, there is a loop C' in G which intersects C transversely in one point. Hence each loop in P which is homotopic to C' must intersect D. Since each loop in F is disjoint with D, C' cannot be in the image of $\pi_1(F)$. But this is a contradiction. So $\pi_1(G) \to \pi_1(P)$ is a monomorphism.

Consequently, $[\pi_1(F) : \pi_1(G)]$ is finite. Let $p: \tilde{P} \to P$ be the finite-fold covering corresponding to $\pi_1(G)$. Let $\tilde{G}$ be a component of $p^{-1}(G)$. Then $\pi_1(\tilde{G}) \to \pi_1(\tilde{P})$ is an isomorphism by construction. In particular, $H_1(\tilde{G}) \to H_1(\tilde{P})$ is an isomorphism. Because $\tilde{P}$ is compact and orientable,

$$\operatorname{rank}(\operatorname{Image}(H_1(\partial\tilde{P}) \to H_1(\tilde{P}))) = \tfrac{1}{2}\operatorname{rank}(H_1(\partial\tilde{P})).$$

Since the components of $p^{-1}(F \cup G)$ are closed, one sees that $p^{-1}(F \cup G)$ must have only two components. So $\tilde{P} = P$ and $\pi_1(G) \to \pi_1(P)$ is an isomorphism. □

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