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A BEST CONSTANT AND THE GAUSSIAN CURVATURE[1]

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ABSTRACT. For axisymmetric $f \in C^\infty(S^2)$ we find conditions to make f the scalar curvature of a metric pointwise conformal to the standard metric of S^2. Closely related to these results, we prove that in the inequality (Moser [**8**])

$$\int_{S^2} e^u \le C e^{\|\nabla u\|_2^2/16\pi} \quad \forall u \in H_1^2(S^2) \text{ with } \int_{S^2} u = 0,$$

the best constant $C = \mathrm{Vol}(S^2)$.

1. Introduction and main results. Given $f(x) \in C^\infty(S^2)$, a very interesting problem is to find a condition on $f(x)$ so that it can be made the scalar curvature function of a metric pointwise conformal to the standard metric of S^2. Assume that $\mathrm{Vol}(S^2) = 4\pi$. This problem is equivalent to the existence of solutions of the equation (cf. [**1, 2**])

$$\Delta u(x) - 2 + f(x)e^{u(x)} = 0, \qquad x \in S^2 \ (\Delta u = \nabla^i \nabla_i u). \tag{1.1}$$

The following results are known (cf. [**3**]): One can solve (1.1) if

(a) $f(x) = f(-x)$ and $f(x) > 0$ somewhere (Moser [**4**]).

(b) $f(x)$ is replaced by $f(\Psi(x))$ for some diffeomorphism Ψ, $f(x) > 0$ somewhere (Kazdan and Warner [**5**]).

(c) $f(x)$ is replaced by $f(x)+h(x)$ for some $h \in \Lambda \triangleq \{\varphi \in C^\infty(S^2) | -\Delta\varphi = \lambda_1\varphi\}$ (Aubin [**6**]).

Kazdan and Warner [**7**] proved that if u is a solution of (1.1), then

$$\int_{S^2} \nabla_i f \nabla^i h \cdot e^u = 0 \quad \forall h \in \Lambda. \tag{1.2}$$

Closely related to the above problems is the following inequality proved by Moser [**8**]:

$$\int_{S^2} e^u \le C e^{\|\nabla u\|_2^2/16\pi} \quad \forall u \in H_1^2(S^2) \text{ with } \int_{S^2} u = 0. \tag{1.3}$$

Let $x = (\vartheta, \varphi) \in S^2$, where $-\pi/2 \le \vartheta \le \pi/2$, $-\pi < \varphi \le \pi$ are the latitude and longitude respectively. Let N, S $\in S^2$ be the North and South poles respectively.

Set

$$C_\vartheta^\infty(S^2) = \{u \in C^\infty(S^2) \mid u \text{ is independent of } \varphi\},$$
$$H_{1\vartheta}^2(S^2) = \text{ the closure of } C_\vartheta^\infty(S^2) \text{ in } H_1^2(S^2).$$

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In §2 we prove the following existence results for equation (1.1):

Consider the functional

$$I(u) = \frac{1}{2}\int_{S^2} |\nabla u|^2 + 2\int_{S^2} u, \qquad u \in H_1^2(S^2), \tag{1.4}$$

and set $\mu = \inf I(u)$ for all $u \in C_\vartheta^\infty(S^2)$ satisfying $\int_{S^2} fe^u = 8\pi$.

THEOREM 1.1. *If $f \in C_\vartheta^\infty(S^2)$ and $\max(f(\mathrm{N}), f(\mathrm{S})) > 0$, then*

$$\mu \leq 8\pi \log \frac{2}{\max(f(\mathrm{N}), f(\mathrm{S}))}. \tag{1.5}$$

If

$$\mu < 8\pi \log \frac{2}{\max(f(\mathrm{N}), f(\mathrm{S}))}, \tag{1.6}$$

then equation (1.1) has a solution $u \in C_\vartheta^\infty(S^2)$.

REMARK 1.1. This result resembles that of Aubin [**2**] for the Yamabe problem and in one of Brezis and Nirenberg [**9**].

EXAMPLE 1.1. (a) $f = \sin\vartheta$. Letting $h = f = \sin\vartheta$ in (1.2), we know that (1.1) has no solution. Hence by Theorem 1.1,

$$\mu = 8\pi \log \frac{2}{\max(f(\mathrm{N}), f(\mathrm{S}))}. \tag{1.7}$$

(b) $f = \sin^2\vartheta$. Since $f(x) = f(-x)$, (1.1) has a solution by Moser [**4**]. In this case, (1.7) holds. Using Theorem 1.5 we have

$$8\pi = \int_{S^2} fe^u \leq \int_{S^2} e^u \leq 4\pi \exp\left\{\frac{1}{16\pi}\int_{S^2} |\nabla u|^2 + \frac{1}{4\pi}\int_{S^2} u\right\};$$

therefore

$$\frac{1}{2}\int_{S^2} |\nabla u|^2 + 2\int_{S^2} u \geq 8\pi \log 2. \tag{1.8}$$

Combining (1.8) and (1.5), we obtain (1.7).

THEOREM 1.2. *Suppose that $f \in C_\vartheta^\infty(S^2)$, $\max(f(\mathrm{N}), f(\mathrm{S})) \leq 0$ and $f(x) > 0$ somewhere. Then equation (1.1) has a solution $u \in C_\vartheta^\infty(S^2)$.*

COROLLARY 1.3. *Suppose that $f \in C_\vartheta^\infty(S^2)$, $f(x) > 0$ somewhere and*

$$\bar{f} \triangleq \frac{1}{\mathrm{Vol}(S^2)}\int_{S^2} f \geq \max(f(\mathrm{N}), f(\mathrm{S})).$$

Then (1.1) has a solution $u \in C_\vartheta^\infty(S^2)$.

COROLLARY 1.4. *If $f \in C_\vartheta^\infty(S^2)$ and at one pole, say N, $f(\mathrm{N}) > 0$, $d^2 f(\mathrm{N})/d\vartheta^2 > 0$ and $f(\mathrm{N}) \geq f(\mathrm{S})$, then (1.1) has a solution $u \in C_\vartheta^\infty(S^2)$.*

The above results are closely related to a best constant. The following theorems are proved in §3.

THEOREM 1.5. *On* (S^2, can) *in* (1.3) *the best* (*smallest possible*) *constant* $C = \mathrm{Vol}(S^2)$; *i.e.*,

$$\int_{S^2} e^u \leq \mathrm{Vol}(S^2) e^{\|\nabla u\|_2^2/16\pi} \quad \forall u \in H_1^2(S^2) \text{ with } \int_{S^2} u = 0. \tag{1.9}$$

Moreover, the equality in (1.9) *holds for*

$$u_\lambda = \log \frac{1-\lambda^2}{(1-\lambda \sin \vartheta)^2} + c_\lambda \quad \forall \lambda \in (-1,1),$$

where

$$c_\lambda = \frac{1}{\lambda} \log \frac{1+\lambda}{1-\lambda} - 2,$$

so $\int_{S^2} u_\lambda = 0$, *and* $c_0 \triangleq \lim_{\lambda \to 0} c_\lambda = 0$.

Consider the functional

$$J(u) = \log \int_{S^2} f e^u - \frac{1}{16\pi} \int_{S^2} |\nabla u|^2 - \frac{1}{4\pi} \int_{S^2} u, \qquad u \in H_1^2(S^2).$$

Moser [8] proved that $J(u)$ is bounded above and stated an open problem. Is $\sup_{u \in H_1^2(S^2); \int f e^u > 0} J(u)$ taken on? Concerning this we have

THEOREM 1.6. $\forall f \in C^\infty(S^2)$ *and* $f(x) > 0$ *somewhere* (*without symmetry assumptions on* f) *we have*

$$\sup_{\substack{u \in H_1^2(S^2) \\ \int f e^u > 0}} J(u) = \log \left(4\pi \max_{x \in S^2} f(x) \right).$$

Moreover, this supremum is never taken on unless $f = \text{const} > 0$; *when* $f = \text{const} > 0$, *this supremum is attained by* $u_\lambda = -2\log(1 - \lambda \sin \vartheta) + C$, $\forall \lambda \in (-1,1)$, $C \in R$.

2. Proofs of existence results. Given $f \in C_\vartheta^\infty(S^2)$, let $\{u_n\}$ be a minimizing sequence in $C_\vartheta^\infty(S^2)$; i.e.,

$$I(u_n) \to \mu \quad \text{and} \quad \int_{S^2} f e^{u_n} = 8\pi \quad \forall n \in N. \tag{2.1}$$

LEMMA 2.1. *If* $\exists C > 0$ *such that*

$$\int_{S^2} |\nabla u_n|^2 \leq C \quad \forall n \in N, \tag{2.2}$$

then equation (1.1) *has a solution* $u \in C_\vartheta^\infty(S^2)$.

PROOF. The same reasoning as in Aubin [2, 5.10(b) and 5.9(b), (c)] shows that

$$u_n \rightharpoonup \bar{u}(H_{1\vartheta}^2(S^2)), \quad \int_{S^2} f e^{\bar{u}} = 8\pi \tag{2.3}$$

and

$$\int_{S^2} \nabla^i \bar{u} \nabla_i h + 2 \int_{S^2} h - a \int_{S^2} f e^{\bar{u}} h = 0 \quad \forall h \in H_{1\vartheta}^2(S^2). \tag{2.4}$$

Since $\bar{u} \in H^2_{1\vartheta}(S^2)$, then by orthogonality of spherical harmonics and noticing that if $v \in C^\infty_\vartheta(S^2)$, then $\Delta v \in C^\infty_\vartheta(S^2)$, $e^v \in C^\infty_\vartheta(S^2)$, one can deduce that (2.4) holds $\forall h \in H^2_1(S^2)$. Thus $u = \bar{u}$ is a weak solution of $\Delta u - 2 + afe^u = 0$. As in Aubin [**2**, 5.9(c)], we get $\bar{u} \in C^\infty_\vartheta(S^2)$. Integrating on S^2 yields $a = 1$.

Thus we are led to find a condition to ensure (2.2), so we can prove that (1.1) has a solution.

LEMMA 2.2. *If $u \in C^\infty_\vartheta(S^2)$ and $\exists 0 < \delta \leq \pi/2$, $c_1, c_2 \in R$, $|\vartheta_0| \leq \pi/2 - \delta$ such that*

$$I(u) \leq c_1 \quad \text{and} \quad u(\vartheta_0) \geq c_2, \tag{2.5}$$

then

$$\int_{S^2} |\nabla u|^2 \leq C(\delta, c_1, c_2), \tag{2.6}$$

where $C(\delta, c_1, c_2)$ depends only on δ, c_1 and c_2.

PROOF. Set

$$M_1 = \{x = (\vartheta, \varphi) \in S^2 | \vartheta_0 \leq \vartheta \leq \pi/2\},$$
$$M_2 = \{x = (\vartheta, \varphi) \in S^2 | -\pi/2 \leq \vartheta \leq \vartheta_0\}.$$

We have the Poincaré inequality

$$\lambda_{1M_i} \int_{M_i} v^2 \leq \int_{M_i} |\nabla v|^2 \quad \forall v \in \mathring{H}^2_1(M_i),\ i = 1, 2, \tag{2.7}$$

where λ_{1M_i} is the first eigenvalue for $-\Delta$ on M_i with $v|_{\partial M_i} = 0$. By a result of Cheeger (cf. [**1**]) $\lambda_{1M_i} \geq \frac{1}{4} h_D(M_i)^2$ and

$$h_D(M_i) = \inf \left\{ \frac{\operatorname{Vol}(\partial\Omega)}{\operatorname{Vol}(\Omega)} \mid \Omega \text{ is a compact subdomain of } M_i \right\}.$$

Thus

$$\lambda_{1M_1} \geq \frac{1}{4} h_D(M_1)^2 = \frac{1}{4} \left(\inf_{-\pi/2+\delta \leq \vartheta_0 \leq \vartheta \leq \pi/2} \frac{2\pi \cos\vartheta}{2\pi(1 - \sin\vartheta)} \right)^2 = c_3(\delta) > 0. \tag{2.8}$$

Similarly,

$$\lambda_{1M_2} \geq c_3(\delta) > 0. \tag{2.8}$$

By (2.7), (2.8), $\forall v \in C^\infty_\vartheta(S^2)$ with $v(\vartheta_0) = 0$ for some ϑ_0, $|\vartheta_0| \leq \pi/2 - \delta$, we have

$$\int_{S^2} |v| = \int_{M_1} |v| + \int_{M_2} |v| \leq c \left[\left(\int_{M_1} v^2 \right)^{1/2} + \left(\int_{M_2} v^2 \right)^{1/2} \right]$$
$$\leq c(\delta) \left(\int_{S^2} |\nabla v|^2 \right)^{1/2}.$$

By (2.5)

$$c_1 \geq \frac{1}{2} \int_{S^2} |\nabla u|^2 + 2 \int_{S^2} u \geq \frac{1}{2} \int_{S^2} |\nabla u|^2 + 8\pi c_2 + 2 \int_{S^2} (u - u(\vartheta_0))$$
$$\geq \frac{1}{2} \int_{S^2} |\nabla u|^2 + 8\pi c_2 - 2 \int_{S^2} |u - u(\vartheta_0)| \geq \frac{1}{2} \int_{S^2} |\nabla u|^2 + 8\pi c_2$$
$$- 2C(\delta) \left(\int_{S^2} |\nabla u|^2 \right)^{1/2}.$$

Therefore (2.6) is true.

REMARK 2.1. More generally, if $u \in C^\infty(S^2)$ (without symmetry assumptions on u) and $\exists c_1, c_2 \in R$ such that $I(u) \le c_1$ and either

(a) $\exists \delta > 0$ such that $\operatorname{meas}\{x \in S^2 \mid u(x) \ge c_2\} \ge \delta$, or

(b) $\exists 0 < \delta \le \pi/2$, $|\vartheta_0| \le \pi/2 - \delta$ such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(\vartheta_0, \varphi)\, d\varphi \ge c_2,$$

then (2.6) holds.

We do not use this result but sketch the proof as follows: In case (a) let $\bar{u}(\vartheta)$ be the symmetric rearrangement for u (cf. [**2**, 2.17]); in case (b) let $\bar{u}(\vartheta) = (1/2\pi)\int_{-\pi}^{\pi} u(\vartheta, \varphi)\, d\varphi$. In both cases one can prove that $\int_{S^2} \bar{u} = \int_{S^2} u$ and $\int_{S^2} |\nabla \bar{u}|^2 \le \int_{S^2} |\nabla u|^2$; hence

(2.9)
$$\begin{aligned}\frac{1}{2}\int_{S^2} |\nabla u|^2 &\le c_1 - 2\int_{S^2} u = c_1 - 2\int_{S^2} \bar{u} \le c_1 - 8\pi c_2 - 2\int_{S^2} (\bar{u} - \bar{u}(\vartheta_0)) \\ &\le c_1 - 8\pi c_2 + 2\int_{S^2} |\bar{u} - \bar{u}(\vartheta_0)| \le c_1 - 8\pi c_2 + c\left(\int_{S^2} |\bar{u} - \bar{u}(\vartheta_0)|^2\right)^{1/2}.\end{aligned}$$

By the proof of Lemma 2.2, we have

$$\left(\int_{S^2} |\bar{u} - \bar{u}(\vartheta_0)|^2\right)^{1/2} \le C(\delta)\left(\int_{S^2} |\nabla \bar{u}|^2\right)^{1/2} \le C(\delta)\left(\int_{S^2} |\nabla u|^2\right)^{1/2}. \tag{2.10}$$

Then (2.6) follows from (2.9) and (2.10).

PROOF OF THEOREM 1.1. Without loss of generality, assume that $f(\mathrm{N}) \ge f(\mathrm{S})$ and $f(\mathrm{N}) > 0$.

(a) Set

$$u_\lambda = \log \frac{2(1-\lambda^2)}{f(\mathrm{N})(1 - \lambda \sin \vartheta)^2}, \qquad \lambda \to 1^-.$$

We have

$$\begin{aligned}\int_{S^2} f e^{u_\lambda} &= \frac{4\pi(1-\lambda^2)}{f(\mathrm{N})}\int_{-\pi/2}^{\pi/2} \frac{f(\vartheta)\cdot \cos\vartheta\, d\vartheta}{(1-\lambda\sin\vartheta)^2} \\ &= 4\pi(1-\lambda^2)\int_{-\pi/2}^{\pi/2} \frac{\cos\vartheta\, d\vartheta}{(1-\lambda\sin\vartheta)^2} \\ &\quad + \frac{4\pi(1-\lambda^2)}{f(\mathrm{N})}\int_{-\pi/2}^{\pi/2} \frac{(f(\vartheta) - f(\mathrm{N}))\cos\vartheta\, d\vartheta}{(1-\lambda\sin\vartheta)^2} \\ &= 8\pi + \frac{4\pi(1-\lambda^2)}{f(\mathrm{N})}\left[\int_{-\pi/2}^{\pi/2-\delta(\varepsilon)} \frac{(f(\vartheta) - f(\mathrm{N}))\cos\vartheta\, d\vartheta}{(1-\lambda\sin\vartheta)^2}\right. \\ &\quad \left. + \int_{\pi/2-\delta(\varepsilon)}^{\pi/2} \frac{(f(\vartheta) - f(\mathrm{N}))\cos\vartheta\, d\vartheta}{(1-\lambda\sin\vartheta)^2}\right] \\ &= 8\pi + \varepsilon(\lambda), \quad \text{where } \varepsilon(\lambda) \to 0 \text{ as } \lambda \to 1^-\end{aligned}$$

and

$$\frac{1}{2}\int_{S^2} |\nabla u_\lambda|^2 + 2\int_{S^2} u_\lambda = 8\pi \log \frac{2}{f(\mathrm{N})}.$$

Therefore (1.5) is true.

(b) If $\mu < 8\pi \log(2/f(\mathrm{N}))$, consider the minimizing sequence $\{u_n\} \subset C^\infty_\vartheta(S^2)$ satisfying (2.1). Notice that $\forall\varepsilon > 0$, $\exists\delta > 0$ such that $f(\vartheta) \le f(\mathrm{N}) + \varepsilon$ if $|\vartheta| \ge \pi/2 - \delta$. Suppose that $\int_{S^2} |\nabla u_n|^2 \to +\infty$ as $n \to +\infty$. Then by Lemma 2.2 we have $u_n(\vartheta) \to -\infty$ uniformly in ϑ for $|\vartheta| \le \pi/2 - \delta$ as $n \to +\infty$. Thus by Theorem 1.5 we have

$$\begin{aligned} 8\pi = \int_{S^2} fe^{u_n} &\le \eta_n + (f(\mathrm{N}) + \varepsilon) \int_{S^2} e^{u_n} \\ &\le \eta_n + (f(\mathrm{N}) + \varepsilon) 4\pi \exp\left\{ \frac{1}{16\pi} \int_{S^2} |\nabla u_n|^2 + \frac{1}{4\pi} \int_{S^2} u_n \right\}, \end{aligned}$$

where

$$\eta_n = \int_{|\vartheta| \le \pi/2 - \delta} fe^{u_n} \to 0.$$

Hence

$$I(u_n) \ge 8\pi \log \frac{8\pi - \eta_n}{4\pi(f(\mathrm{N}) + \varepsilon)}.$$

Since η_n and $\varepsilon > 0$ can be arbitrarily small, we get $\mu \ge 8\pi \log(2/f(\mathrm{N}))$, which contradicts (1.6). Therefore there exists a subsequence, still denoted by $\{u_n\}$, such that $\int_{S^2} |\nabla u_n|^2 \le C$. Then by Lemma 2.1, (1.1) has a solution $u \in C^\infty_\vartheta(S^2)$.

PROOF OF THEOREM 1.2.

Case 1. $\max(f(\mathrm{N}), f(\mathrm{S})) < 0$. By the continuity of f, $\exists\delta > 0$ such that $f(\vartheta) \le 0$ if $|\vartheta| \ge \pi/2 - \delta$. Consider the minimizing sequence $\{u_n\}$ as above; we have

$$8\pi = \int_{S^2} fe^{u_n} \le \int_{|\vartheta| \le \pi/2 - \delta} fe^{u_n} \le 4\pi \max_{x \in S^2} f(x) \cdot \exp\{\max_{|\vartheta| \le \pi/2 - \delta} u_n\};$$

i.e.,

$$\max_{|\vartheta| \le \pi/2 - \delta} u_n(\vartheta) \ge \log \frac{2}{\max f}.$$

Thus, by Lemmas 2.2 and 2.1 we obtain a solution $u \in C^\infty_\vartheta(S^2)$ of (1.1).

Case 2. $\max(f(\mathrm{N}), f(\mathrm{S})) = 0$. Again, consider the minimizing sequence $\{u_n\} \subset C^\infty_\vartheta(S^2)$. Assuming that $\int_{S^2} |\nabla u_n|^2 \to +\infty$ as $n \to +\infty$, we proceed as in the proof of Theorem 1.1(b) and get $I(u_n) \to +\infty$, a contradiction. Then by Lemma 2.1, (1.1) has a solution $u \in C^\infty_\vartheta(S^2)$.

PROOF OF COROLLARY 1.3.

Case 1. $\max(f(\mathrm{N}), f(\mathrm{S})) \le 0$. Corollary 1.3 follows from Theorem 1.2.

Case 2. $\bar{f} > \max(f(\mathrm{N}), f(\mathrm{S})) > 0$. Set $w = \log(2/\bar{f})$. Then $\int_{S^2} fe^w = 8\pi$ and

$$\mu \le \frac{1}{2} \int_{S^2} |\nabla w|^2 + 2 \int_{S^2} w = 8\pi \log \frac{2}{\bar{f}} < 8\pi \log \frac{2}{\max(f(\mathrm{N}), f(\mathrm{S}))}.$$

Thus Corollary 1.3 follows from Theorem 1.1.

Case 3. $\bar{f} = \max(f(\mathrm{N}), f(\mathrm{S})) > 0$. If (1.1) has no solution, then by Theorem 1.1

$$\mu = 8\pi \log \frac{2}{\max(f(\mathrm{N}), f(\mathrm{S}))} = 8\pi \log \frac{2}{\bar{f}}.$$

But $w = \log(2/\bar{f})$ achieves the infimum μ, so we obtain a contradiction.

Consider the diffeomorphism $F_{p,\lambda}\colon S^2 \to S^2$, $\lambda \in (-1,1)$, $p \in S^2$, defined as follows: Suppose that $p = \mathrm{N}$. Then if $x = (\vartheta, \varphi) \in S^2$, $y = F_{N,\lambda}(x) = (\vartheta_1, \varphi_1) \in S^2$, we have

$$\sin\vartheta_1 = \frac{\sin\vartheta - \lambda}{1 - \lambda\sin\vartheta}, \qquad \varphi_1 = \varphi. \tag{2.11}$$

Then $F_{p,-\lambda} \circ F_{p,\lambda} = \mathrm{id}\ \forall p \in S^2,\ \lambda \in (-1,1)$.

LEMMA 2.3 (CF. AUBIN [**2**]). *$\forall p \in S^2$, $\lambda \in (-1,1)$, equation* (1.1) *has a solution for $f \in C^\infty(S^2)$ (without symmetry assumptions on f) if and only if* (1.1) *has a solution for $f \circ F_{p,\lambda}$.*

PROOF. This lemma can be seen by the composition of two conformal transformations of S^2, the first one $F_{p,\lambda}$ has its pole at p (cf. [**2**, the proof of Corollary 5.12]). We can also prove this lemma as follows:

Assume that $p = N$. If $v(y)$ is a solution of (1.1)—i.e., $\Delta_y v(y) - 2 + f(y)e^{v(y)} = 0$—set

$$u(x) = \log\frac{1-\lambda^2}{(1-\lambda\sin\vartheta)^2} + v(F_{\mathrm{N},\lambda}(x)), \qquad \lambda \in (-1,1)$$

(see (2.11)). We have

$$\cos\vartheta_1 = \frac{\sqrt{1-\lambda^2}\cos\vartheta}{1-\lambda\sin\vartheta}, \quad \frac{d\vartheta_1}{d\vartheta} = \frac{\sqrt{1-\lambda^2}}{1-\lambda\sin\vartheta},$$

$$\Delta_x u = \frac{1}{\cos\vartheta}\frac{\partial}{\partial\vartheta}\left(\cos\vartheta\frac{\partial u}{\partial\vartheta}\right) + \frac{1}{\cos^2\vartheta}\frac{\partial^2 u}{\partial\varphi^2}.$$

Direct computation shows that $u(x)$ satisfies $\Delta_x u - 2 + f(F_{\mathrm{N},\lambda}(x))e^u = 0$, and since $F_{\mathrm{N},-\lambda} \circ F_{\mathrm{N},\lambda} = \mathrm{id}$, Lemma 2.3 follows.

PROOF OF COROLLARY 1.4. Use the conformal diffeomorphism $F_{\mathrm{N},\lambda}$ (see (2.11)) and set $f_\lambda = f \circ F_{\mathrm{N},\lambda}$. Then

$$\bar f_\lambda = \frac{1}{2}\int_{-\pi/2}^{\pi/2} f_\lambda(\vartheta)\cos\vartheta\,d\vartheta = \frac{1-\lambda^2}{2}\int_{-\pi/2}^{\pi/2}\frac{f(\vartheta_1)\cos\vartheta_1\,d\vartheta_1}{(1+\lambda\sin\vartheta_1)^2}.$$

Since $d^2 f(\mathrm{N})/d\vartheta^2 > 0$ and $df(\mathrm{N})/d\vartheta = 0$, $\exists\alpha > 0,\ \varepsilon > 0$ such that $f(\vartheta_1) - f(\mathrm{N}) \geq a(\pi/2 - \vartheta_1)^2$ for $0 \leq \pi/2 - \vartheta_1 \leq \varepsilon$. Thus

$$\begin{aligned}\bar f_\lambda - f(\mathrm{N}) &= \frac{1-\lambda^2}{2}\int_{-\pi/2}^{\pi/2}\frac{(f(\vartheta_1) - f(\mathrm{N}))\cos\vartheta_1\,d\vartheta_1}{(1+\lambda\sin\vartheta_1)^2} \\ &= \frac{1-\lambda^2}{2}\left(\int_{-\pi/2}^{\pi/2-\varepsilon} + \int_{\pi/2-\varepsilon}^{\pi/2}\right).\end{aligned}$$

Let ε be fixed and $\lambda \to -1^+$. Then one can verify that $\int_{-\pi/2}^{\pi/2-\varepsilon} \leq c(\varepsilon)$ and $\int_{\pi/2-\varepsilon}^{\pi/2} \to +\infty$. Therefore $\bar f_\lambda - f(\mathrm{N}) > 0$ if λ is sufficiently close to -1. Then applying Corollary 1.3 for f_λ and Lemma 2.3, we deduce Corollary 1.4.

REMARK 2.2. (a) Corollary 1.4 can also be derived directly from Theorem 1.1 without invoking Lemma 2.3.

(b) Combining Lemma 2.3 with the previous results, by composition with $F_{p,\lambda}$, one can find a class of functions f for which (1.1) has no solution and $\nabla_i f \nabla^i h$

changes its sign on S^2 for all $h \in \Lambda \triangleq \{\varphi \in C^\infty(S^2) | -\Delta\varphi = \lambda_1\varphi\}$; also one can find a class of functions f for which (1.1) has a solution and f is neither axisymmetric nor antipodally symmetric.

3. A best constant.

LEMMA 3.1. *If* $u \in C^\infty_\vartheta(S^2)$ *satisfies* $\Delta u - 2 + 2e^u = 0$, *then*

$$u = \log\frac{1-\lambda^2}{(1-\lambda\sin\vartheta)^2}, \qquad -1 < \lambda < 1. \tag{3.1}$$

PROOF. u satisfies

$$\frac{1}{\cos\vartheta}(\cos\vartheta \cdot u_\vartheta)_\vartheta - 2 + 2e^u = 0, \qquad \vartheta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \ \left(u_\vartheta \triangleq \frac{du}{d\vartheta}\right). \tag{3.2}$$

Let $r = 2\tan(\pi/4 + \vartheta/2)$ (stereographic projection). (3.2) becomes

$$u_{rr} + \frac{1}{r}u_r - 2\left(1 + \frac{r^2}{4}\right)^{-2} + 2\left(1+\frac{r^2}{4}\right)^{-2} e^u = 0, \qquad r \in (0,\infty). \tag{3.3}$$

Set

$$u = \log((4+r^2)^2/32) + y. \tag{3.4}$$

(3.3) is reduced to

$$ry_{rr} + y_r + re^y = 0, \qquad r \in (0,\infty). \tag{3.5}$$

Now (3.5) can be solved as follows (cf. [**10**, 6.76]): Setting

$$w(t) = ry_r, \qquad t = r^2e^y, \tag{3.6}$$

from (3.5), we obtain $(w+2)w_t + 1 = 0$. Thus

$$(w+2)^2 + 2t = 4a^2, \qquad a > 0 \text{ a constant}; \tag{3.7}$$

i.e.,

$$r^2y_r^2 + 4ry_r + 2r^2e^y + 4(1-a^2) = 0. \tag{3.8}$$

Combining (3.5) with (3.8), we get

$$2r^2y_{rr} = r^2y_r^2 + 2ry_r + 4(1-a^2). \tag{3.9}$$

Setting $y = -2\log p$, we get from (3.9) the Euler equation

$$r^2p_{rr} - rp_r + (1-a^2)p = 0.$$

Hence $p = c_1r^{1+a} + c_2r^{1-a} > 0$ and

$$y = -2\log(c_1r^{1+a} + c_2r^{1-a}), \qquad r \in (0,\infty). \tag{3.10}$$

Substituting (3.10) into (3.6) then into (3.7), we get $c_1c_2 = 1/8a^2$; obviously $c_1 > 0$, $c_2 > 0$. By (3.4), (3.10) and $c_1c_2 = 1/8a^2$ we obtain

$$u = 2\log\frac{c_3a(r^2+4)}{2(c_3^2r^{1+a} + r^{1-a})}, \qquad c_3 \triangleq \sqrt{8}ac_1 > 0,\ a > 0. \tag{3.11}$$

Since $u \in C^\infty_\vartheta(S^2)$ and $r = 2\tan(\pi/4 + \vartheta/2)$, $\lim_{r\to0} u$ and $\lim_{r\to+\infty} u$ must be finite. From (3.11) we get $a = 1$. Setting $\lambda = (1-4c_3^2)/(1+4c_3^2)$, we obtain (3.1). One can verify that $u \in C^\infty_\vartheta(S^2)$ and $\Delta u - 2 + 2e^u = 0$.

LEMMA 3.2. *We have* (*see* (1.4))

$$\beta \triangleq \inf_{\substack{u \in H_1^2(S^2) \\ \int_{S^2} e^u = 4\pi}} I(u) = 0. \tag{3.12}$$

The infimum is attained by

$$u_\lambda = \log \frac{1-\lambda^2}{(1-\lambda \sin\vartheta)^2}, \qquad -1 < \lambda < 1.$$

PROOF. (a) Direct evaluation shows that $\int_{S^2} e^{u_\lambda} = 4\pi$ and $I(u_\lambda) = 0$; therefore $\beta \le 0$.

(b) If $\beta < 0$, since $C^\infty(S^2)$ is dense in $H_1^2(S^2)$ and the mapping $H_1^2(S^2) \ni \varphi \to e^\varphi \in L_1(S^2)$ is compact [**2**, Theorem 2.46], then $\exists v_0 \in C^\infty(S^2)$ such that $\int_{S^2} e^{v_0} = 4\pi$ and $I(v_0) < 0$. Set $b = \min_{x \in S^2} v_0(x)$.

(3.13)

$$C_b \triangleq \inf I(v) \quad \text{for } v \in H_1^2(S^2) \text{ satisfying } \int_{S^2} e^v = 4\pi \text{ and } \operatorname*{ess\,inf}_{x \in S^2} v(x) \ge b.$$

Then $C_b < 0$. Let $\{v_i\}_{i=1}^\infty \subset H_1^2(S^2)$ be a sequence satisfying (3.13) and $I(v_i) \to C_b < 0$. Using Friedrich's mollifier for v_i, then using symmetric rearrangement [**2**, 2.16, 2.17], we find a sequence, still denoted by $\{v_i\} \subset H_{1\vartheta}^2(S^2)$, satisfying (3.13), $I(v_i) \to C_b < 0$ and $v_i(\vartheta)$ is nondecreasing in ϑ.

(c) From $I(v_i) \to C_b$ and $\int_{S^2} v_i \ge 4\pi b$, we get that $\int_{S^2} |\nabla v_i|^2$ is bounded. As in the proof of Lemma 2.1, we have $v_i \rightharpoonup w \in H_{1\vartheta}^2(S^2)$,

$$\int_{S^2} e^w = 4\pi, \qquad \operatorname*{ess\,inf}_{x \in S^2} w(x) \ge b \quad \text{and} \quad I(w) = C_b < 0. \tag{3.14}$$

Moreover, $w(\vartheta)$ is nondecreasing in ϑ. We claim that $\operatorname{ess\,inf} w(x) = b$ and $w(x) \not\equiv b$. In fact, if $w(x) \equiv b = \text{const}$, since $\int_{S^2} e^w = 4\pi$, we have $w(x) \equiv b = 0$ and $I(w) = 0$, which contradicts (3.14); if $\operatorname{ess\,inf} w(x) > b$, then one can derive that $w \in H_{1\vartheta}^2(S^2)$ satisfies $\Delta w(x) - 2 + \lambda e^{w(x)} = 0$, $x \in S^2$, and the Lagrange multiplier $\lambda = 2$, as in [**2**], $w \in C_\vartheta^\infty(S^2)$ and by Lemma 3.1, $w = \log[(1-\lambda^2)/(1-\lambda\sin\vartheta)^2]$, again we get $I(w) = 0$, a contradiction. Hence $\exists -\pi/2 \le \vartheta_0 < \pi/2$ such that $w(\vartheta) \equiv b$ if $\vartheta \le \vartheta_0$ and $w(\vartheta) > b$ if $\vartheta_0 < \vartheta \le \pi/2$.

(d) Set

$$M_\tau = \{x = (\vartheta, \varphi) \in S^2 | \vartheta > \tau\}.$$

From (c), noticing that if $h \in C_0^\infty(M_{\vartheta_0})$ satisfies $\int_{S^2} e^w h = 0$ then $\int_{S^2} \nabla^i w \nabla_i h + 2\int_{S^2} h = 0$, one can derive that w satisfies

$$\Delta w(x) - 2 + \rho e^{w(x)} = 0 \quad \text{for } x \in M_{\vartheta_0}, \tag{3.15}$$

where $\rho \in R$ is the Lagrange multiplier. Since $e^w \in L_p$, $\forall p > 1$ and $w \in H_{1\vartheta}^2(S^2)$, by the standard regularity argument we get $w \in C^\infty(M_{\vartheta_0})$. We claim that $\rho > 0$. In fact, if $\rho \le 0$, integrating (3.15) on M_ϑ ($\vartheta > \vartheta_0$) shows that $w(\vartheta)$ is strictly decreasing in ϑ if $\vartheta > \vartheta_0$, a contradiction.

(e) Since $w \in H^2_{1\vartheta}(S^2)$, one can prove that $w \in C(S^2 \backslash \{\mathrm{N} \cup \mathrm{S}\})$. Using the proof of Lemma 3.1, we have

$$w = 2\log \frac{c_3 a(r^2+4)}{2(c_3^2 r^{1+a} + r^{1-a})} + c_4, \qquad a > 0,\ c_3 > 0,$$
$$r = 2\tan\left(\frac{\pi}{4} + \frac{\vartheta}{2}\right), \qquad \vartheta \geq \vartheta_0.$$

We claim that $a = 1$. Otherwise we have $\lim_{\vartheta \to \pi/2} w = \pm\infty$, a contradiction. Thus, $\exists \lambda \in (-1, 1)$ such that

$$w = \begin{cases} 2\log \dfrac{1-\lambda \sin\vartheta_0}{1-\lambda\sin\vartheta} + b & \text{if } \vartheta > \vartheta_0, \\ b & \text{if } \vartheta \leq \vartheta_0. \end{cases} \tag{3.16}$$

(f) From (3.16) and $\int_{S^2} e^w = 4\pi$ we get

$$2e^{-b} = \frac{2(1-\lambda\sin\vartheta_0)}{1-\lambda} - \frac{\lambda\cos^2\vartheta_0}{1-\lambda} \triangleq 2p - q, \tag{3.17}$$

where

$$p \triangleq \frac{1-\lambda\sin\vartheta_0}{1-\lambda} > 0, \quad q \triangleq \frac{\lambda\cos^2\vartheta_0}{1-\lambda}.$$

From (3.16) we have

$$I(w) = \frac{1}{2}\int_{S^2} |\nabla w|^2 + 2\int_{S^2} w = \frac{-2q}{p} + 4\log p + 4b. \tag{3.18}$$

By (3.17), (3.18) we have

$$I(w) = \frac{4e^{-b}}{p} + 4\log p + 4(b-1) \triangleq F(p), \qquad p > 0.$$

Minimizing the function $F(p)$ for $0 < p < +\infty$, we obtain $I(w) \geq 0$, contradicting (3.14) and thus completing the proof.

PROOF OF THEOREM 1.5. By Lemma 3.2, the best constant C in (1.3) is

$$C = \sup_{\substack{u \in H_1^2(S^2) \\ \int_{S^2} u = 0}} \frac{\int_{S^2} e^u}{e^{\|\nabla u\|_2^2/16\pi}} = \sup_{u \in H_1^2(S^2)} \frac{\int_{S^2} e^u}{\exp\{\|\nabla u\|_2^2/16\pi + (1/4\pi)\int_{S^2} u\}}$$
$$= \frac{4\pi}{\inf_{u \in H_1^2(S^2); \int_{S^2} e^u = 4\pi} e^{I(u)/8\pi}} = 4\pi,$$

and, in general, $C = \mathrm{Vol}(S^2)$.

The rest of Theorem 1.5 can be verified by direct evaluation.

PROOF OF THEOREM 1.6.

Case a. $f \neq \text{const}$. By Theorem 1.5, we have

$$\begin{aligned} J(u) &< \log \max_{x \in S^2} f(x) + \log \int_{S^2} e^u - \frac{1}{16\pi}\int_{S^2} |\nabla u|^2 - \frac{1}{4\pi}\int_{S^2} u \\ &\leq \log(4\pi \max_{x \in S^2} f(x)). \end{aligned} \tag{3.19}$$

Without loss of generality, suppose that $f(\mathrm{N}) = \max_{x\in S^2} f(x)$. Set

$$v_\lambda = \log \frac{1-\lambda^2}{4\pi(1-\lambda\sin\vartheta)^2}.$$

One can verify that $e^{v_\lambda} \to \delta_N$ (Dirac δ-function) as $\lambda \to 1^-$ and $I(v_\lambda) = -8\pi\log 4\pi$. Thus $J(v_\lambda) \to \log(4\pi \max_{x\in S^2} f(x))$ as $\lambda \to 1^-$.

Case b. $f(x) \equiv a > 0$, where a is a constant. Similarly we get $J(u) \le \log(4\pi a)$ and $J(u_\lambda) = \log(4\pi a)$, where $u_\lambda = -2\log(1-\lambda\sin\vartheta) + C$, $\lambda \in (-1,1)$, $C \in R$. Thus Theorem 1.6 follows.

Theorem 1.6 tells us that if we want to prove the existence of solutions of (1.1) via critical points of a functional, generally speaking, we have to look for a local extremum, a saddle point and so on.

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