PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 97, Number 4, August 1986

WHITNEY LEVELS IN $C_p(X)$ ARE ARS

MARK LYNCH[1]

ABSTRACT. For X a metric continuum, and $p \in X$, we show that the Whitney levels in the relative hyperspace $C_p(X) = \{K \in C(X) | p \in K\}$ are absolute retracts.

1. Introduction. For (X, d) a metric continuum, let $C(X)$ denote the hyperspace of subcontinua with the Hausdorff metric H. A *Whitney map* $\mu: C(X) \to [0,1]$ is a map such that $\mu(\{x\}) = 0$ for each $x \in X$, $\mu(X) = 1$, and $\mu(K) < \mu(M)$ whenever $K \subsetneqq M$. Such maps may always be constructed [**8**]. The point-inverses $\mu^{-1}(t)$, $0 \le t \le 1$, are subcontinua in $C(X)$ [**4**], and are called *Whitney levels*. For $p \in X$, let $C_p(X) = \{K \in C(X) | p \in K\}$ be the relative hyperspace, and let μ_p denote the restriction of μ to $C_p(X)$. Eberhart [**3**] showed that $C_p(X)$ is always an AR. In this paper we consider the relative Whitney levels $\mu_p^{-1}(t)$ in $C_p(X)$. We will use a construction in a space of order arcs to show that each Whitney level $\mu_p^{-1}(t)$ is an AR.

Krasinkiewicz and Nadler [**7**] and Rogers [**11**] have shown that $\mu_p^{-1}(t)$ is arcwise connected, and Rogers [**10**] showed that $\mu_p^{-1}(t)$ is acyclic. We point out that the Whitney levels $\mu^{-1}(t)$ need not be arcwise connected [**6**], and even if X is a 2-cell, they need not be ARs [**9**].

2. The space of orders arcs $\Lambda_p(X)$. An arc $\alpha \subset C(X)$ is an *order arc* if, for all $K, M \in \alpha$, either $K \subset M$ or $M \subset K$. Then $\bigcap \alpha = \bigcap \{M | M \in \alpha\}$ and $\bigcup \alpha = \bigcup \{M | M \in \alpha\}$ are the endpoints of α. For every pair $A, B \in C(X)$ with $A \subset B$, there exists an order arc α with $\bigcap \alpha = A$ and $\bigcup \alpha = B$ [**6**]. An order arc α may be parametrized by defining $\alpha(t)$ to be the unique $K \in \alpha$ such that $\mu(K) = (1-t) \cdot \mu(\bigcap \alpha) + t \cdot \mu(\bigcup \alpha)$. Let Λ_p denote the space of *maximal order arcs* in $C_p(X)$, i.e., $\Lambda_p = \{\text{order arcs } \alpha \subset C_p(X) | \alpha(0) = \{p\} \text{ and } \alpha(1) = X\} \subset C(C(X))$. For each $t \in [0,1]$, the *evaluation map* $e_t: \Lambda_p \to \mu_p^{-1}(t)$, defined by $e_t(\alpha) = \alpha(t)$, is onto. We will show that $\mu_p^{-1}(t)$ is an absolute extensor for metric spaces by imitating Dugundji's proof of the extension property for maps into locally convex linear metric spaces [**2**]. To do this, we utilize the evaluation map e_t and a type of convex structure on Λ_p.

For $\alpha_1, \ldots, \alpha_n \in \Lambda_p$ and $t_1, \ldots, t_{n-1} \in [0,1]$, let $\beta = \langle \alpha_1, t_1; \ldots; \alpha_{n-1}, t_{n-1}; \alpha_n \rangle$ be the element of Λ_p defined by

$$\beta = \{\alpha_1(s) | 0 \le s \le t_1\} \cup \{\alpha_1(t_1) \cup \alpha_2(s) | 0 \le s \le t_2\}$$
$$\cup \cdots \cup \{\alpha_1(t_1) \cup \cdots \cup \alpha_{n-1}(t_{n-1}) \cup \alpha_n(s) | 0 \le s \le 1\}.$$

Received by the editors July 24, 1985.
1980 *Mathematics Subject Classification*. Primary 54B20, 54C55; Secondary 54F20.
Key words and phrases. Absolute retract, hyperspace of subcontinua, Whitney level.
[1]This work represents part of the author's dissertation, written at Louisiana State University under the supervision of Professor Doug Curtis.

This construction is continuous, in the sense that if $\alpha_i^m \to \alpha_i$ and $t_i^m \to t_i$ as $m \to \infty$, for each i, then $\langle \alpha_1^m, t_1^m; \dots; \alpha_{n-1}^m, t_{n-1}^m; \alpha_n^m \rangle \to \langle \alpha_1, t_1; \dots; \alpha_{n-1}, t_{n-1}; \alpha_n \rangle$.

3. Extending maps into $\mu_p^{-1}(t)$. Let (Z, ρ) be a metric space, and $A \subset Z$ a closed subset. Given a map $g: A \to \mu_p^{-1}(t)$, we define an extension $\hat{g}: Z \to \mu_p^{-1}(t)$ of g by a Dugundji-type construction. For every $x \in Z - A$, let $B_x = \{z \in Z | \rho(x, z) < \frac{1}{2} \cdot \rho(x, A)\}$. Let $\mathbf{U} = \{U_\alpha | \alpha \in \mathbf{A}\}$ be a neighborhood finite open refinement of $\{B_x | x \in Z - A\}$, indexed by a well-ordered set $\mathbf{A}$. Let $\{\phi_\alpha | \alpha \in \mathbf{A}\}$ be a partition of unity of $Z - A$ subordinated to $\mathbf{U}$. With each $\alpha \in \mathbf{A}$, associate $a_\alpha \in A$ as follows: Choose $x_\alpha \in U_\alpha$, and take $a_\alpha \in A$ with $\rho(x_\alpha, a_\alpha) < 2 \cdot \rho(x_\alpha, A)$. For each $\alpha \in \mathbf{A}$, choose $\beta_\alpha \in \Lambda_p$ such that $e_t(\beta_\alpha) = \beta_\alpha(t) = g(a_\alpha)$. Then with each $x \in Z - A$ there is associated a finite set $\{\beta_\alpha | \phi_\alpha(x) > 0\}$ of elements of Λ_p. The extension $\hat{g}: Z \to \mu_p^{-1}(t)$ is defined in the following steps:

(1) For $x \in Z - A$, let $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ be the ordering in $\mathbf{A}$ of those elements α for which $\phi_\alpha(x) > 0$, and define

$$\tau(x, \alpha_i) = \frac{\phi_{\alpha_i}(x)}{\phi_{\alpha_i}(x) + \cdots + \phi_{\alpha_n}(x)}, \qquad i = 1, 2, \dots, n.$$

Note that $\tau(x, \alpha_n) = 1$.

(2) With $\beta_{\alpha_1}, \dots, \beta_{\alpha_n}$ the elements of Λ_p corresponding to x, define $\beta_x = \langle \beta_{\alpha_1}, \tau(x, \alpha_1); \dots; \beta_{\alpha_{n-1}}, \tau(x, \alpha_{n-1}); \beta_{\alpha_n} \rangle$.

(3) Define $\hat{g}(x) = e_t(\beta_x) = \beta_x(t)$.

CLAIM 1. $\hat{g}$ is continuous on $Z - A$.

Consider $x \in Z - A$, with $\alpha_1 < \cdots < \alpha_n$ as in step (1) above. For any $y \in Z$ sufficiently close to x, each $\phi_{\alpha_i}(y)$ will be near $\phi_{\alpha_i}(x)$, $i = 1, \dots, n$. Clearly, this implies that if $\gamma_1 < \cdots < \gamma_{n+m}$ is the ordering in $\mathbf{A}$ of $\{\alpha | \phi_\alpha(y) > 0\}$, then for each k such that $\gamma_k = \alpha_i$ for some i, $\tau(y, \gamma_k)$ is near $\tau(x, \alpha_i)$, and for all other γ_k, either $\tau(y, \gamma_k)$ is near 0 or $\alpha_n < \gamma_k$. It follows that β_y is near β_x, and $\hat{g}(y)$ is near $\hat{g}(x)$.

CLAIM 2. $\hat{g}$ is continuous on $\operatorname{bd} A$.

Consider $a \in \operatorname{bd} A$, and let $y \in Z - A$ denote a point near a. Let $\{\alpha_1, \dots, \alpha_n\} = \{\alpha \in \mathbf{A} | \phi_\alpha(y) > 0\}$. Then for each $i \le n$, the point $a_{\alpha_i} \in A$ associated with α_i is near a, thus $\beta_{\alpha_i}(t) = g(a_{\alpha_i})$ is near $g(a)$. Note that the construction of the order arc β_y forces $\beta_y(t) \subset \beta_{\alpha_1}(t) \cup \beta_{\alpha_2}(t) \cup \cdots \cup \beta_{\alpha_n}(t)$. Since $M = \bigcup_1^n \beta_{\alpha_i}(t)$ is an element of $C_p(X)$ near $g(a) \in \mu_p^{-1}(t)$, $\mu(M)$ is near $t = \mu(\beta_y(t))$. Since $\beta_y(t) \subset M$, the nature of the Whitney map μ forces $\beta_y(t)$ and M to be close. Then $\hat{g}(y) = \beta_y(t)$ is near $\hat{g}(a) = g(a)$.

Thus, the extension $\hat{g}: Z \to \mu_p^{-1}(t)$ of g is continuous, and this concludes the proof of our main result:

THEOREM. *Each Whitney level $\mu_p^{-1}(t)$ in $C_p(X)$ is an AR.*

There are several easy corollaries.

COROLLARY 1. *$\{X\}$ is an unstable point in $C_p(X)$.*

PROOF. Given $\varepsilon > 0$, choose $t < 1$ sufficiently close to 1 so that

$$\operatorname{diam}_H(\mu_p^{-1}([t, 1])) < \varepsilon.$$

Since $\mu_p^{-1}(t)$ is an AR, there is a retraction $r: \mu_p^{-1}([t,1]) \to \mu_p^{-1}(t)$. Then r extends by the identity to a retraction $R: C_p(X) \to \mu_p^{-1}([0,t])$. We have $R(C_p(X)) \subset C_p(X) - \{X\}$ and $H(R(M), M) < \varepsilon$ for each $M \in C_p(X)$. Thus, $\{X\}$ is unstable in $C_p(X)$.

Note that the above retraction R shows that $\mu_p^{-1}([0,t])$ is an AR. In fact, a similar argument shows that $\mu_p^{-1}([s,t])$ is an AR for all $s < t$.

COROLLARY 2. *$C_p(X) - \{X\}$ is an AR.*

PROOF. Since $C_p(X) - \{X\}$ is an ANR, it suffices to show that it is n-connected for all n [**5**]. Let $f: S^n \to C_p(X) - \{X\}$ be a map of the n-sphere. Choose $t < 1$ such that $f(S^n) \subset \mu_p^{-1}([0,t])$. Then f is null-homotopic in the AR $\mu_p^{-1}([0,t])$, thus $C_p(X) - \{X\}$ is n-connected for all n.

If X has a cut point p, then clearly $\mu^{-1}(t) = \mu_p^{-1}(t)$ for all t in some neighborhood of 1. Thus we have the following

COROLLARY 3. *If X has a cut point, then*
(i) *for all t in some neighborhood of* 1, *$\mu^{-1}(t)$ is an AR*; *and*
(ii) *$\{X\}$ is unstable in $C(X)$.*

The general question suggested by part (ii) has been answered in [**1**]: For X a Peano continuum, $\{X\}$ is stable in $C(X)$ if and only if X is a finite graph with no cut points.

BIBLIOGRAPHY

1. D. W. Curtis, *Stable points in hyperspaces of Peano continua*, preprint.
2. J. Dugundji, *An extension of Tietze's theorem*, Pacific J. Math. **1** (1951), 353–367.
3. Carl Eberhart, *Intervals of continua which are Hilbert cubes*, Proc. Amer. Math. Soc. **68** (1978), 220–224.
4. Carl Eberhart and Sam B. Nadler, Jr., *The dimension of certain hyperspaces*, Bull. Polon. Acad. Sci. Math. **19** (1971), 1027–1034.
5. S. T. Hu, *Theory of retracts*, Wayne State Univ. Press, 1965.
6. J. L. Kelley, *Hyperspaces of a continuum*, Trans. Amer. Math. Soc. **52** (1942), 22–36.
7. J. Krasinkiewicz and Sam B. Nadler, Jr., *Whitney properties*, Fund. Math. **98** (1978), 165–180.
8. Sam B. Nadler, Jr., *Hyperspaces of sets*, Dekker, New York and Basel, 1978.
9. Ann Petrus, *Whitney maps and Whitney properties in $C(X)$*, Topology Proc., vol. 1 (Proc. 1976 Topology Conference, Auburn Univ.), Math. Dept., Auburn Univ., Auburn, Ala., 1977, pp. 147–172.
10. James T. Rogers, Jr., *Applications of a Vietoris-Begle theorem for multi-valued maps to the cohomology of hyperspaces*, Michigan J. Math. **22** (1976), 315–319.
11. ——, *Whitney continua in the hyperspace $C(X)$*, Pacific J. Math. **58** (1975), 569–584.

DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LOUISIANA 70803

Current address: 5466 Harpers Farm Road, Apartment A4, Columbia, Maryland 21044