

SHORTER NOTES

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A SIMPLIFIED PROOF OF HEINZ INEQUALITY AND SCRUTINY OF ITS EQUALITY

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Dedicated to Professor Masanori Fukamiya in celebration of his having been specially decorated by the Japanese government for his distinguished achievements in mathematics.

ABSTRACT. An operator means a bounded linear operator on a Hilbert space H . We give a simplified proof of the following inequality:

$$(I_1) \quad |(Tx, y)|^2 \leq (|T|^{2\alpha}x, x)(|T^*|^{2(1-\alpha)}y, y)$$

for any operator T and for any $x, y \in H$ and for any real number α with $0 \leq \alpha \leq 1$. In case $0 < \alpha < 1$, the equality in (I_1) holds iff $|T|^{2\alpha}x$ and T^*y are linearly dependent iff Tx and $|T^*|^{2(1-\alpha)}y$ are linearly dependent. (I_1) is equivalent to

$$(I_2) \quad |(Tx, y)| \leq \| |T|^\alpha x \| \| |T^*|^{1-\alpha} y \|,$$

so one might believe that the equality in (I_1) or (I_2) would hold iff $|T|^{2\alpha}x$ and $|T^*|^{2(1-\alpha)}y$ are linearly dependent or iff $|T|^\alpha x$ and $|T^*|^{1-\alpha}y$ are linearly dependent, but we can give counterexamples to these mistakes. By this fact, the form of (I_1) is more convenient than (I_2) in order to remind us of the case when the equality in (I_1) or (I_2) holds.

1. Statement of the results. An operator means a bounded linear operator on a Hilbert space.

THEOREM 1. For any operator T on a Hilbert space H ,

$$(I_1) \quad |(Tx, y)|^2 \leq (|T|^{2\alpha}x, x)(|T^*|^{2(1-\alpha)}y, y)$$

holds for any $x, y \in H$ and for any real number α with $0 \leq \alpha \leq 1$.

(i) $0 < \alpha < 1$. The equality in (I_1) holds iff $|T|^{2\alpha}x$ and T^*y are linearly dependent iff Tx and $|T^*|^{2(1-\alpha)}y$ are linearly dependent.

(ii) $\alpha = 1$. The equality in (I_1) holds iff Tx and y are linearly dependent.

(iii) $\alpha = 0$. The equality in (I_1) holds iff x and T^*y are linearly dependent.

PROOF. In case $\alpha = 1$ or 0 , the result is obvious, so we assume $0 < \alpha < 1$.

Received by the editors March 1, 1985 and, in revised form, July 16, 1985.

1980 *Mathematics Subject Classification.* Primary 47A30; Secondary 47A99.

Key words and phrases. Heinz inequality, polar decomposition.

Proof of inequality (I₁). Let $T = U|T|$ be the polar decomposition of T where U means the partial isometry and $|T| = (T^*T)^{1/2}$ with $N(U) = N(|T|)$ where $N(S)$ means the kernel of an operator S . First of all, we state the following obvious but important relation:

(*) $N(S^q) = N(S)$ for any positive operator S and for any positive number q .

Next we show that, for any positive number q ,

$$(1) \quad |T^*|^q = U|T|^qU^*.$$

As (*) holds for $|T|$ and U^*U is the initial projection, we have $\overline{R(|T|^q)} = \overline{R(|T|)}$, so $U^*U|T|^q = |T|^q$. And $|T^*|^2 = TT^* = U|T||T|U^* = U|T|U^*U|T|U^* = (U|T|U^*)^2$, so that $|T^*| = U|T|U^*$, since $U|T|U^*$ is positive. By induction, $|T^*|^{n/m} = U|T|^{n/m}U^*$ holds for any natural numbers m and n ; then letting $n/m \rightarrow q$, we have $|T^*|^q = U|T|^qU^*$, so we have (1). Put $\beta = 1 - \alpha$. By (1) we have

$$\begin{aligned} |(Tx, y)|^2 &= |(U|T|x, y)|^2 = |(|T|x, U^*y)|^2 \\ &= |(|T|^{\alpha}x, |T|^{\beta}U^*y)|^2 \leq \| |T|^{\alpha}x \|^2 \| |T|^{\beta}U^*y \|^2 \\ &= (|T|^{2\alpha}x, x)(U|T|^{2\beta}U^*y, y) = (|T|^{2\alpha}x, x)(|T^*|^{2\beta}y, y), \end{aligned}$$

so the proof of inequality (I₁) is complete.

Scrutiny of the equality in (I₁). The equality in the inequality above holds iff $|T|^{\alpha}x$ and $|T|^{\beta}U^*y$ are linearly dependent iff $|T|^{2\alpha}x$ and $|T|U^*y$ are linearly dependent by (*) for $|T|$ iff

(2) $|T|^{2\alpha}x$ and T^*y are linearly dependent.

On the other hand, the equality holds iff $|T|^{\alpha}x$ and $|T|^{\beta}U^*y$ are linearly dependent iff $|T|x$ and $|T|^{2\beta}U^*y$ are linearly dependent by (*) for $|T|$; equivalently $U|T|x$ and $U|T|^{2\beta}U^*y$ are linearly dependent by (*) for $|T|$ and $N(U) = N(|T|)$ iff

(3) Tx and $|T^*|^{2\beta}y$ are linearly dependent

by (1), so that (2) holds iff (3) holds. Hence the equality in (I₁) holds iff (2) holds iff (3) holds, so the proof is complete.

Inequality (I₁) in Theorem 1 may be called the “*weighted mixed Schwarz inequality*”. Theorem 1 implies the following famous Heinz inequality since $A \geq B \geq 0$ implies $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$.

THEOREM A [1–3]. *Let T be an operator. If A and B are positive operators such that $\|Tx\| \leq \|Ax\|$ and $\|T^*y\| \leq \|By\|$ for all $x, y \in H$. Then*

$$|(Tx, y)| \leq \|A^\alpha x\| \|B^{1-\alpha}y\| \quad \text{for } 0 \leq \alpha \leq 1.$$

Theorem 1 easily implies Corollary 1 as follows:

COROLLARY 1. *For any operator T on a Hilbert space H ,*

$$|(Tx, y)|^2 \leq (|T|x, x)(|T^*|y, y)$$

*holds for any $x, y \in H$. The equality holds iff $|T|x$ and T^*y are linearly dependent iff Tx and $|T^*|y$ are linearly dependent.*

REMARK. One might believe that the equality in Theorem 1 would hold iff $|T|^{2\alpha}x$ and $|T^*|^{2(1-\alpha)}y$ are linearly dependent. But here we can give a counterexample. Let $T = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\alpha = \frac{1}{2}$. Then $|T^*|y = 2|T|x$, but $|(Tx, y)|^2 = 36 \neq (|T|x, x)(|T^*|y, y) = 54$. In case $0 < \alpha < 1$, we have to emphasize that the equality in Theorem 1 holds iff $|T|^{2\alpha}x$ and T^*y are linearly dependent

iff Tx and $|T^*|^{2(1-\alpha)}y$ are linearly dependent. By this fact (I_1) in Theorem 1 is more convenient than $|(Tx, y)| \leq \| |T|^\alpha x \| \| |T^*|^{1-\alpha} y \|$ which is equivalent to (I_1) because it reminds us of the case when the equality in (I_1) holds. Also we can give an example such that the equality in Theorem 1 does not always hold even if $|T|^\alpha x$ and $|T^*|^{(1-\alpha)}y$ are linearly dependent. Results in this paper would remain valid for unbounded operators under slightly modifications.

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