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SHORTER NOTES

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A SIMPLIFIED PROOF OF HEINZ INEQUALITY AND SCRUTINY OF ITS EQUALITY

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Dedicated to Professor Masanori Fukamiya in celebration of his having been specially decorated by the Japanese government for his distinguished achievements in mathematics.

ABSTRACT. An operator means a bounded linear operator on a Hilbert space H. We give a simplified proof of the following inequality:

$$(\mathrm{I}_1) \qquad |(Tx,y)|^2 \le (|T|^{2\alpha}x,x)(|T^*|^{2(1-\alpha)}y,y)$$

for any operator T and for any $x, y \in H$ and for any real number α with $0 \le \alpha \le 1$. *In case $0 < \alpha < 1$, the equality in* (I_1) *holds iff $|T|^{2\alpha}x$ and T^*y are linearly dependent iff Tx and $|T^*|^{2(1-\alpha)}y$ are linearly dependent.* (I_1) is equivalent to

$$(\mathrm{I}_2) \qquad |(Tx,y)| \le \| \, |T|^{\alpha}x\| \, \| \, |T^*|^{1-\alpha}y\|,$$

so one might believe that the equality in (I_1) or (I_2) would hold iff $|T|^{2\alpha}x$ and $|T^*|^{2(1-\alpha)}y$ are linearly dependent or iff $|T|^{\alpha}x$ and $|T^*|^{1-\alpha}y$ are linearly dependent, but we can give counterexamples to these mistakes. By this fact, the form of (I_1) is more convenient than (I_2) in order to remind us of the case when the equality in (I_1) or (I_2) holds.

1. Statement of the results. An operator means a bounded linear operator on a Hilbert space.

THEOREM 1. *For any operator T on a Hilbert space H,*

$$(\mathrm{I}_1) \qquad |(Tx,y)|^2 \le (|T|^{2\alpha}x,x)(|T^*|^{2(1-\alpha)}y,y)$$

holds for any $x, y \in H$ and for any real number α with $0 \le \alpha \le 1$.

(i) $0 < \alpha < 1$. *The equality in* (I_1) *holds iff $|T|^{2\alpha}x$ and T^*y are linearly dependent iff Tx and $|T^*|^{2(1-\alpha)}y$ are linearly dependent.*

(ii) $\alpha = 1$. *The equality in* (I_1) *holds iff Tx and y are linearly dependent.*

(iii) $\alpha = 0$. *The equality in* (I_1) *holds iff x and T^*y are linearly dependent.*

PROOF. In case $\alpha = 1$ or 0, the result is obvious, so we assume $0 < \alpha < 1$.

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Proof of inequality (I_1). Let $T = U|T|$ be the polar decomposition of T where U means the partial isometry and $|T| = (T^*T)^{1/2}$ with $N(U) = N(|T|)$ where $N(S)$ means the kernel of an operator S. First of all, we state the following obvious but important relation:

(∗) $N(S^q) = N(S)$ *for any positive operator* S *and for any positive number* q.

Next we show that, for any positive number q,

$$|T^*|^q = U|T|^qU^*. \tag{1}$$

As (∗) holds for $|T|$ and U^*U is the initial projection, we have $\overline{R(|T|^q)} = \overline{R(|T|)}$, so $U^*U|T|^q = |T|^q$. And $|T^*|^2 = TT^* = U|T|\,|T|U^* = U|T|U^*U|T|U^* = (U|T|U^*)^2$, so that $|T^*| = U|T|U^*$, since $U|T|U^*$ is positive. By induction, $|T^*|^{n/m} = U|T|^{n/m}U^*$ holds for any natural numbers m and n; then letting $n/m \to q$, we have $|T^*|^q = U|T|^qU^*$, so we have (1). Put $\beta = 1 - \alpha$. By (1) we have

$$\begin{aligned}|(Tx,y)|^2 &= |(U|T|x,y)|^2 = |(|T|x,U^*y)|^2\\ &= |(|T|^\alpha x, |T|^\beta U^*y)|^2 \le \|\,|T|^\alpha x\|^2\|\,|T|^\beta U^*y\|^2\\ &= (|T|^{2\alpha}x,x)(U|T|^{2\beta}U^*y,y) = (|T|^{2\alpha}x,x)(|T^*|^{2\beta}y,y),\end{aligned}$$

so the proof of inequality (I_1) is complete.

Scrutiny of the equality in (I_1). The equality in the inequality above holds iff $|T|^\alpha x$ and $|T|^\beta U^*y$ are linearly dependent iff $|T|^{2\alpha}x$ and $|T|U^*y$ are linearly dependent by (∗) for $|T|$ iff

(2) $|T|^{2\alpha}x$ *and* T^*y *are linearly dependent.*

On the other hand, the equality holds iff $|T|^\alpha x$ and $|T|^\beta U^*y$ are linearly dependent iff $|T|x$ and $|T|^{2\beta}U^*y$ are linearly dependent by (∗) for $|T|$; equivalently $U|T|x$ and $U|T|^{2\beta}U^*y$ are linearly dependent by (∗) for $|T|$ and $N(U) = N(|T|)$ iff

(3) Tx *and* $|T^*|^{2\beta}y$ *are linearly dependent*

by (1), so that (2) holds iff (3) holds. Hence the equality in (I_1) holds iff (2) holds iff (3) holds, so the proof is complete.

Inequality (I_1) in Theorem 1 may be called the "*weighted mixed Schwarz inequality*". Theorem 1 implies the following famous Heinz inequality since $A \ge B \ge 0$ implies $A^\alpha \ge B^\alpha$ for any $\alpha \in [0,1]$.

THEOREM A [**1–3**]. *Let T be an operator. If A and B are positive operators such that $\|Tx\| \le \|Ax\|$ and $\|T^*y\| \le \|By\|$ for all $x, y \in H$. Then*

$$|(Tx,y)| \le \|A^\alpha x\|\,\|B^{1-\alpha}y\| \quad \textit{for } 0 \le \alpha \le 1.$$

Theorem 1 easily implies Corollary 1 as follows:

COROLLARY 1. *For any operator T on a Hilbert space H,*

$$|(Tx,y)|^2 \le (|T|x,x)(|T^*|y,y)$$

holds for any $x, y \in H$. The equality holds iff $|T|x$ and T^*y are linearly dependent iff Tx and $|T^*|y$ are linearly dependent.

REMARK. One might believe that the equality in Theorem 1 would hold iff $|T|^{2\alpha}x$ and $|T^*|^{2(1-\alpha)}y$ are linearly dependent. But here we can give a counterexample. Let $T = \left(\begin{smallmatrix}0&2\\1&0\end{smallmatrix}\right)$, $x = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$, $y = \left(\begin{smallmatrix}1\\4\end{smallmatrix}\right)$ and $\alpha = \frac{1}{2}$. Then $|T^*|y = 2|T|x$, but $|(Tx,y)|^2 = 36 \ne (|T|x,x)(|T^*|y,y) = 54$. *In case $0 < \alpha < 1$, we have to emphasize that the equality in Theorem 1 holds iff $|T|^{2\alpha}x$ and T^*y are linearly dependent* *iff* Tx *and* $|T^*|^{2(1-\alpha)}y$ *are linearly dependent.* By this fact (I_1) in Theorem 1 is more convenient than $|(Tx,y)| \leq \| |T|^\alpha x\| \| |T^*|^{1-\alpha}y\|$ which is equivalent to (I_1) because it reminds us of the case when the equality in (I_1) holds. Also we can give an example such that the equality in Theorem 1 does not always hold even if $|T|^\alpha x$ and $|T^*|^{(1-\alpha)}y$ are linearly dependent. Results in this paper would remain valid for unbounded operators under slightly modifications.

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