PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 98, Number 1, September 1986

EXTENDING VALUATIONS TO FINITE DIMENSIONAL DIVISION ALGEBRAS

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ABSTRACT. Let D be a division algebra finite dimensional over its center F. It is shown that a (Krull) valuation v on F extends to a valuation on D iff v extends uniquely to each commutative field K with $F \subseteq K \subseteq D$.

A valuation on a division ring is a function v: $D^* \to G$ (where G is a totally ordered group) satisfying, for all $a, b \in D^* = D - \{0\}$,

(i) $v(ab) = v(a) + v(b)$;

(ii) $v(a + b) \geqslant \min\{v(a), v(b)\}$ if $b \neq -a$.

It is convenient to use additive notation for G, though G is not assumed to be abelian. If E is any subring of D, v restricts to a valuation on E, and we say that (D, v) is an *extension* of (E, v). The standard reference for valuations on division rings is Schilling's book [**S**].

One of the fundamental features of commutative valuation theory is that for fields (meaning commutative fields) $F \subseteq K$, every valuation on F has an extension to K (possibly many different extensions). However, if we replace the fields F and K by division rings this extension property does not always hold. This failure is a major obstacle in noncommutative valuation theory. On the positive side, it is well known that if F is a field with Henselian valuation v, then v extends to a valuation on any F-division algebra D finite dimensional over F.

We generalize this Henselian result to give a precise criterion for when a valuation on a field F extends to one on a given finite dimensional F-central division algebra. This criterion was proved by P. M. Cohn in [**C**, Theorem 1] for valuations v on F with value group $v(F^*)$ a subgroup of the additive group of real numbers, i.e., when the valuation ring of (F, v) has Krull dimension 1. Cohn's argument used completions, and does not appear to generalize readily to higher Krull dimensions. However, there are significant examples of valued division algebras with rings of Krull dimension at least 2. Notably, such algebras have been used in the construction of noncrossed product algebras (cf. [**A**, **T**, **JW**]) and also in the examples of division algebras D with $SK_1(D)$ nontrivial (cf. [**P**, **DK**]). So, it is worthwhile to see that the extension criterion holds without any Krull dimension restriction.

Received by the editors August 19, 1985.
1980 *Mathematics Subject Classification*. Primary 16A39; Secondary 12E15.
[1]Supported in part by the N.S.F.

THEOREM. *Let D be a division ring finite dimensional over its center F, and let v be a valuation on F. Then v extends to a valuation on D if and only if v has a* unique *extension to each field K with $F \subseteq K \subseteq D$.*

PROOF. Let $\Gamma_F := v(F^*)$ be the value group of v, a torsion-free abelian group; let Δ be the divisible hull of Γ_F (so $\Delta \cong \Gamma_F \otimes_{\mathbf{Z}} \mathbf{Q}$). The total ordering on Γ_F extends uniquely to a total ordering on Δ, and for each field L algebraic over F and each extension w of v to L we may view $w(L^*)$ as a subgroup of Δ.

Suppose v extends uniquely to each field K, $F \subseteq K \subseteq D$. Define a function w: $D^* \to \Delta$ by

$$(*) \qquad w(a) = (1/n)v(\mathrm{Nrd}(a)),$$

where $n = \sqrt{\dim_F D} \in \mathbf{Z}$ and Nrd is the reduced norm from D to F (as described, e.g., in [**R**, §9]). Clearly the restriction $w|_F$ of w to F coincides with v. We will show that w is a valuation on D. First we verify that for every maximal subfield K of D, $w|_K$ is the valuation on K extending v. Let M be the normal closure of K over F, and let u: $M^* \to \Delta$ be any valuation on M extending v on F. For any $b \in K^*$ $\mathrm{Nrd}(b) = N_{K/F}(b) = b_1 \cdots b_n$, where $N_{K/F}$ is the norm from K to F and each $b_i \in M$ is a conjugate of b over F. For each i there is an F-automorphism σ_i of M with $\sigma_i(b) = b_i$. Since $u|_K$ and $(u \circ \sigma_i)|_K$ are each valuations on K extending v, by hypothesis they must agree. So, $u(b_i) = u(\sigma_i(b)) = u(b)$. Hence,

$$\begin{aligned} w(b) &= \frac{1}{n}v(\mathrm{Nrd}(b)) = \frac{1}{n}v(b_1 \cdots b_n) \\ &= \frac{1}{n}(u(b_1) + \cdots + u(b_n)) = u(b). \end{aligned}$$

Thus, $w|_K = u|_K$, which is the valuation on K extending v.

To see that w is a valuation on all of D, take any $a, b \in D^*$. Because Nrd is multiplicative, we have $w(ab) = w(a) + w(b)$. Assume $b \neq -a$, and let K be any maximal subfield of D containing $a^{-1}b$. Since $w|_K$ is a valuation, $w(1 + a^{-1}b) \geqslant \min\{w(1), w(a^{-1}b)\}$. Thus, using the multiplicative property for w,

$$\begin{aligned} w(a + b) &= w(a) + w(1 + a^{-1}b) \geqslant w(a) + \min\{w(1), w(a^{-1}b)\} \\ &= \min\{w(a), w(b)\}, \end{aligned}$$

as desired.

For the converse, suppose v extends to a valuation w on D. We first check that the totally ordered value group Γ_D of w is abelian. For, if $\alpha + \beta + (-\alpha) < \beta$ in Γ_D, then $\alpha + k\beta + (-\alpha) < k\beta$ for all natural numbers k. But for some k, $k\beta \in \Gamma_F$ since D is algebraic over F. Because Γ_F is a central subgroup of Γ_D, $\alpha + k\beta + (-\alpha) = k\beta$, a contradiction. Thus, Γ_D is abelian, and we have $w(aba^{-1}) = w(b)$ for all $a, b \in D^*$.

Let $W = \{a \in D^* \mid w(a) \geqslant 0\} \cup \{0\}$ be the valuation ring of w and let V be the valuation ring of v. For any $a \in W$, let $f(x) = x^k + c_{k-1}x^{k-1} + \cdots + c_0 \in F[x]$ be the minimal polynomial of a over F. By an old theorem of Wedderburn [**W**, Lemma 4 or **D**, pp. 230–231] f factors completely in $D[x]$,

$$f(x) = (x - a_1)(x - a_2) \cdots (x - a_k),$$

where each $a_i = b_i a b_i^{-1}$ for some $b_i \in D^*$. So, $w(a_i) = w(a) \geqslant 0$; hence the coefficients of f lie in $W \cap F = V$. That is, every a in W is integral over V.

Let K be any field, $F \subseteq K \subseteq D$. Then $W_0 := W \cap K$ is a valuation ring of K, hence integrally closed, and we just saw that W_0 is integral over V. It is well known (see, e.g., [**B**, §8, No. 6, Proposition 6]) that the valuation rings of K extending V are in 1-1 correspondence with the maximal ideals of the integral closure of V in K. That integral closure is W_0, which is local. Thus, the valuation of W_0 (i.e., $w|_K$) is the unique extension of v to K, and the Theorem is proved.

Note the following interesting consequences of the Theorem and its proof:

COROLLARY. *Let D, F, and v be as in the Theorem, and suppose v extends to a valuation w on D. Then*:

(i) *w is the only extension of v to D; in fact, w is determined from v by formula* (*) *above.*

(ii) *The valuation ring of w equals $\{a \in D \mid a$ is integral over $V\}$, which coincides with $\{a \in D \mid \mathrm{Nrd}(a) \in V\}$, where V is the valuation ring of v.*

Observe that the result on division algebras over Henselian valued fields is immediate from the Theorem. A valuation v on a field is Henselian iff v extends uniquely to each field K algebraic over F. The Theorem indicates why most work in noncommutative valuation theory has been carried out over Henselian fields.

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