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# CLASS GROUPS, TOTALLY POSITIVE UNITS, AND SQUARES

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ABSTRACT. Given a totally real algebraic number field $K$, we investigate when totally positive units, $U_K^+$, are squares, $U_K^2$. In particular, we prove that the rank of $U_K^+/U_K^2$ is bounded above by the minimum of (1) the 2-rank of the narrow class group of $K$ and (2) the rank of $U_L^+/U_L^2$ as $L$ ranges over all (finite) totally real extension fields of $K$. Several applications are also provided.

**1. Notation and preliminaries.** Let $K$ be an algebraic number field and let $C_K$ denote the ideal class group in the ordinary or "wide" sense. Let $C_K^{(+)}$ denote the "narrow" ideal class group of $K$. Thus $|C_K| = h_K$, the "wide" class number of $K$, and $|C_K^{(+)}| = h_K^{(+)}$, the "narrow" class number of $K$. We denote the Hilbert class field of $K$ by $K^{(1)}$; i.e., $\mathrm{Gal}(K^{(1)}/K) \cong C_K$, and we denote the "narrow" Hilbert class field by $K^{(+)}$; i.e., $\mathrm{Gal}(K^{(+)}/K) \cong C_K^{(+)}$. Moreover we adopt the "bar" convention to mean "modulo squares"; for example, $\overline{C}_K = C_K/C_K^2$.

Let $U_K$ denote the group of units of the ring of algebraic integers of $K$. When $K$ is totally real, we let $U_K^+$ denote the subgroup of totally positive units; i.e., those units $u$ such that $u^\sigma > 0$ for all embeddings $\sigma$ of $K$ into $\mathbf{R}$. Finally, for any finite abelian group $A$ with $|\overline{A}| = 2^d$, $d$ is called the 2-rank of $A$, which we denote by $\dim_2 A$.

**2. Results.** We are concerned with the question:

(*) When is $U_K^+ = U_K^2$?

We begin by observing that $\dim_2(\overline{U}_K^+) = 0$ if and only if $K^{(+)} = K^{(1)}$ [**6**, Theorem 3.1, p. 203]. In particular, when $K$ is a real finite Galois extension of 2-power degree over $Q$, then $\dim_2(\overline{U}_K^+) = 0$ if and only if $N(U_K) = \{\pm 1\}$ [**3**, Theorem 1, p. 166]. For example, when $K$ is a real quadratic field, then $\dim_2(\overline{U}_K^+) = 0$ if and only if the norm of the fundamental unit is $-1$. Necessary and sufficient conditions (in terms of the arithmetic of the underlying quadratic field $K$) for the existence of a fundamental unit of norm $-1$ are unknown (see [**8**]). This indicates the difficulty of solving (*) for the simplest even degree case. In this regard one may ask whether (*) is equivalent to such a norm statement for other fields. In a recent letter to the authors, V. Ennola answered (*) for cyclic cubic fields $K$ as follows: Let $\varepsilon$ be a norm positive unit of $K$ such that $-1$ and the conjugates of $\varepsilon$ generate the unit group. Then $\dim_2(\overline{U}_K^+) = 0$ if and only if $\varepsilon$ is *not* totally positive. However, as with

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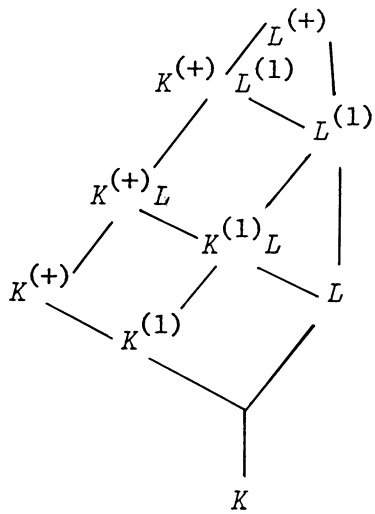
the quadratic field case, the latter does not readily translate into arithmetic conditions on the underlying cyclic cubic field $K$. We have not been able to verify that such a norm condition holds for a larger class of fields. For example, it would be interesting to investigate this question for quartic fields. However, we do have the following result which gives an upper bound on $\overline{U}_K^+$ in terms of $\overline{U}_L^+$ for a totally real extension field $L$ of $K$. For example, this will allow us to translate the norm criterion from a quadratic field to any of its totally real number field extensions. We note that the following generalizes [**3**, Theorem 2, p. 168].

THEOREM 2.1. *Suppose $Q \subseteq K \subseteq L$ with $L$ totally real and finite over $Q$. Then* $\dim_2 \overline{U}_K^+ \leqslant \dim_2 \overline{U}_L^+$.

PROOF. First we show that $K^{(1)} \subseteq L^{(1)}$ and $K^{(+)} \subseteq L^{(+)}$. Both $K^{(1)}L/L$ and $K^{(+)}L/L$ are Galois extensions with abelian Galois groups, since $\mathrm{Gal}(K^{(1)}L/L) \cong \mathrm{Gal}(K^{(1)}/(K^{(1)} \cap L))$, which is a subgroup of the class group $C_K = \mathrm{Gal}(K^{(1)}/K)$, and $\mathrm{Gal}(K^{(+)}L/L) \cong \mathrm{Gal}(K^{(+)}/K^{(+)} \cap L)$, which is a subgroup of the narrow class group $C_K^{(+)} = \mathrm{Gal}(K^{(+)}/K)$. By [**1**] we have that *all* $L$-primes are unramified in $K^{(1)}L$ and all *finite* $L$-primes are unramified in $K^{(+)}L$. It follows that $K^{(1)} \subseteq K^{(1)}L \subseteq L^{(1)}$ and $K^{(+)} \subseteq K^{(+)}L \subseteq L^{(+)}$.

We show next that $K^{(+)} \cap L^{(1)} = K^{(1)}$. The inclusion $K^{(1)} \subseteq K^{(+)} \cap L^{(1)}$ is clear. Since $K \subseteq K^{(+)} \cap L^{(1)} \subseteq K^{(+)}$, we see that $K^{(+)} \cap L^{(1)}$ is an abelian extension of $K$ and that every finite $K$-prime is unramified in $K^{(+)} \cap L^{(1)}$. Moreover, since $L$ is totally real, then $L^{(1)}$ is totally real, and since $K^{(+)} \cap L^{(1)} \subseteq L^{(1)}$, then all infinite $K^{(+)} \cap L^{(1)}$-primes are real. Hence $K^{(+)} \cap L^{(1)} \subseteq K^{(1)}$, and we have shown that $K^{(+)} \cap L^{(1)} = K^{(1)}$.

The following diagram illustrates our situation.




Since $K^{(+)} \cap L^{(1)} = K^{(1)}$ and $K^{(+)}/K^{(1)}$ is Galois, the extensions $K^{(+)}/K^{(1)}$ and $L^{(1)}/K^{(1)}$ are linearly disjoint, and $|K^{(+)}L^{(1)}:L^{(1)}| = |K^{(+)}:K^{(1)}|$. Thus $2^{\dim_2 \overline{U}_K^+} = |K^{(+)}:K^{(1)}| = |K^{(+)}L^{(1)}:L^{(1)}|$ divides $|L^{(+)}:L^{(1)}| = 2^{\dim_2 \overline{U}_L^+}$, and so $\dim_2 \overline{U}_K^+ \leqslant \dim_2 \overline{U}_L^+$. Q.E.D.

With $L$ and $K$ as in Theorem 2.1 we have the following

COROLLARY 2.2. *Let $p$ be a prime and let $K$ be a subfield of $L = Q(\zeta_p + \zeta_p^{-1})$. If the class number of $Q(\zeta_p)$ is odd, then* $\dim_2 \overline{U}_K^+ = \dim_2 \overline{U}_L^+ = 0$.

PROOF. A classical result of Kummer (for example see [**8**, p. 128]) is that $\dim_2 \overline{U}_L^+ = 0$ whenever the class number of $Q(\zeta_p)$ is odd. Therefore the result now follows from Theorem 2.1. Q.E.D.

It is worth noting at this juncture that as a result of recent work of Shimura [**10**] Kummer's classical result cited in the proof of Corollary 2.2 is now extended to $\dim_2(\overline{U}_L^+) = 0$ if and only if the class number of $Q(\zeta_p)$ is odd. Here $L = Q(\zeta_p + \zeta_p^{-1})$ as in Corollary 2.2.

Furthermore, by [**3**, p. 175], if $L = Q(\zeta_f + \zeta_f^{-1})$, where $f$ is composite, then $\dim_2 \overline{U}_L^+ > 0$, and $N(U_L) = \{1\}$.

An application of Theorem 2.1 is the following

EXAMPLE 2.3. Let $K = Q(\alpha)$ where the minimal polynomial of $\alpha$ over $Q$ is $x^3 - x^2 - 234x + 729$. Then $K$ has conductor 703; i.e., the minimal cyclotomic field containing $K$ is $Q(\zeta_{703})$, so $K \subseteq L = Q(\zeta_{703} + \zeta_{703}^{-1})$. It can be shown that $\dim_2 \overline{U}_K^+ = 2$. Thus by Theorem 2.1 $\dim_2 \overline{U}_L^+ \geqslant 2$. For a list of such examples of cyclic cubic fields the reader may consult [2].

Next we prove

PROPOSITION 2.4. *Let $K$ be a totally real algebraic number field. Then* $\dim_2 \overline{U}_K^+ \leqslant \dim_2 C_K^{(+)}$.

PROOF. From [**6**, Theorem 3.1, p. 203] we have that

$$\dim_2\left(\overline{U}_K^+\right) = \dim_2 \mathrm{Gal}\left(K^{(+)}/K^{(1)}\right) \leqslant \dim_2 \mathrm{Gal}\left(K^{(+)}/K\right) = \dim_2\left(C_K^+\right). \quad \text{Q.E.D.}$$

We note that, in general, it is not correct to claim that $\dim_2 \overline{U}_K^+ \leqslant \dim_2 C_K$. For example, if $K = Q(\sqrt{3})$, then $\dim_2 C_K = 0$. However, $K(\sqrt{-1})$ is unramified except at the real primes of $K$, and in fact $K^{(+)} = K(\sqrt{-1})$ [**6**, Theorem 3.10, p. 210]. Thus $\dim_2(\overline{U}_K^+) = \dim_2 C_K^{(+)} = 1$. However, under more restrictive hypotheses it is possible to achieve $\dim_2 \overline{U}_K^+ \leqslant \dim_2 C_K$.

THEOREM 2.5 (ORIAT [**9**]). *Let $K$ be a finite real Galois extension of $Q$ with Galois group of odd exponent $n$ and suppose that $-1$ is congruent to a power of 2 modulo $n$. Then* $\dim_2 \overline{U}_K^+ \leqslant \dim_2 C_K$. *Moreover* $\dim_2 C_K^{(+)} = \dim_2 C_K$.

In particular, when $h_K$ is odd, we have $\dim_2 \overline{U}_K^+ = 0$. (See [**5**] for an independent proof of this fact, distinct from that of [**9**]. The authors of [**5**] were unaware of the existence of [**9**] at the time of that writing.)

In the presence of Proposition 2.4, $\dim_2 \overline{U}_K^+ \leqslant \dim_2 C_K$ is of course an immediate consequence of $\dim_2 C_K = \dim_2 C_K^{(+)}$.

We now exhibit a proof of Theorem 2.5 in the case where $K$ is abelian. This is a simple proof based on the self-duality results established in [**11**]. Before so doing we need to set the stage with some additional notation and concepts.

Let $O_K$ denote the ring of integers of an algebraic number field $K$. We define $\alpha \in O_K$ to be *singular* if the ideal $(\alpha)$ is the square of an ideal. Moreover $\alpha$ is called *odd* if $(\alpha)$ is relatively prime to (2). Furthermore, $\alpha$ is called *primary* if it is odd and $\alpha \equiv \zeta^2 \pmod 4$ for some $\zeta \in O_K$. (For details and further development on these concepts we refer the reader to Hecke [**4**, pp. 217–237].) We denote by $O_K^0$ the singular primary numbers and note that by [**4**, Theorem 120, p. 137] we have $\mu \in O_K^0$ with $\mu \notin O_K^2$ if and only if $K \subsetneq K(\sqrt{\mu}) \subseteq K_2^{(+)}$. It follows that $|K_2^{(+)}:K| = |\overline{O}_K^0|$, where $K_2^{(+)}$ is the maximal abelian extension of $K$ unramified at the finite primes such that $\mathrm{Gal}(K_2^{(+)}/K)$ is a direct sum of copies of $\mathbf{Z}/2\mathbf{Z}$. Thus by Kummer theory we have that $K_2^{(+)}$ is the maximal subfield of $K^{(+)}$ of the form $K(\sqrt{\mu_1}, \ldots, \sqrt{\mu_r})$ for some $\mu_i \in K$, $i = 1, 2, \ldots, r$. Similarly, define $K_2^{(1)}$ as the maximal abelian extension of $K$ unramified at *all* primes with $\mathrm{Gal}(K_2^{(1)}/K)$ being a direct sum of copies of $\mathbf{Z}/2\mathbf{Z}$. Therefore $K_2^{(1)}$ is the maximal subfield of $K^{(1)}$ of the form $K(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_s})$ for some $\alpha_i \in K$, $i = 1, 2, \ldots, s$. Furthermore, we note that $K_2^{(1)} = K_2^{(+)} \cap K^{(1)}$.

Now if $O_K^+$ denotes the subgroup of $O_K$ consisting of totally positive integers, then $\mu \in O_K^+ \cap O_K^0$ with $\mu \notin O_K^2$ if and only if $K \subsetneq K(\sqrt{\mu}) \subseteq K_2^{(1)}$, from which it follows that $|K_2^{(1)}:K| = |\overline{O}_K^+|$.

Now we are in a position to prove Theorem 2.5 under the assumption that $K$ is a finite real extension of $Q$ with abelian Galois group $G$.

PROOF. Let $M$ be a simple $F_2G$-module where $F_2$ is the field of two elements. Then by Schur's lemma $M \cong F_2Ge$ for some idempotent $e$. Now let $\psi$ be the standard involution of $F_2G$ given by $\psi(g) = g^{-1}$ for all $g \in G$. Then by the same argument as in the proof of [**5**, Theorem, p. 615] we have that $F_2Ge \cong F_2G\psi(e)$, resulting from $-1$ being a power of 2 modulo $n$. Hence we have shown that all simple $F_2G$-modules are self-dual. Therefore from [**11**, Corollary 1, p. 157] we have that $\dim_2 \overline{U}_K^+ \leqslant \dim_2 C_K$.

By the self-duality established above, we may use exactly the same reasoning as used by Taylor on $\overline{U}_K^+$ and $\overline{U}_K^0$ in [**11**, (*), p. 157] to establish $\overline{O}_K^+ \cong \overline{O}_K^0$. Hence by the discussion preceding the proof, we have $K_2^{(1)} = K_2^{(+)}$; i.e., $\dim_2 C_K = \dim_2 C_K^{(+)}$. Q.E.D.

In what follows, the *signature map* from $U_K$ to $F_2G$ is defined by $\mathrm{sgn}(u) = \sum_{\sigma \in G} s(u^\sigma)$, where $s$ is called the *signature* of $u$ with $s$: $K^* \to F_2$ defined by $s(k) = 0$ if $k > 0$ and $s(k) = 1$ if $k < 0$.

It is interesting to note that Lagarias [**7**] has proved the equivalence of

(2.7) $\dim_2 C_K = \dim_2 C_K^{(+)}$.

(2.8) All odd singular integers $\alpha$ have their signature type determined by the congruence class of $\alpha$ modulo 4.

(2.9) There are $\alpha$ of all signature types with $\alpha$ an odd singular integer.

(2.10) $K_2^{(+)}$ is totally real.

Thus, it would be of interest to investigate those totally real algebraic number fields $K$ for which the Sylow 2-subgroup of $\mathrm{Gal}(K^{(+)}/K)$ is elementary abelian. It is not enough to know that the Sylow 2-subgroup of $\mathrm{Gal}(K^{(1)}/K)$ is elementary abelian. For example in [**2**] we see that "most" of the Sylow 2-subgroups of $\mathrm{Gal}(K^{(1)}/K)$ are elementary abelian where $K$ is a cyclic cubic field. However, there

are no instances known to the authors in the cyclic cubic case where $\mathrm{Gal}(K^{(+)}/K)$ has elementary abelian Sylow 2-subgroup.

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