GROWTH OF HARMONIC CONJUGATES IN THE UNIT DISC

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Abstract. Assuming some mild regularity conditions on a positive nondecreasing
function \( \psi(x) = O(x^a) \) (for some \( a > 0, x \to \infty \)), we show that
\[
M_p(r, u) = O\left( \psi\left( \frac{1}{1-r} \right) \right) \quad (r \to 1, 0 < p < 1)
\]
implies \( M_p(r, v) = O(\psi^p(1/(1-r)))^{1/p}, \) where \( u(z) + iv(z) \) is holomorphic in the
open unit disc and
\[
\psi^p(x) = \int_{1/2}^x \frac{\psi^p(t)}{t} dt, \quad x \geq \frac{1}{2}.
\]

1. Introduction. Throughout this note \( \psi \) will denote a positive nondecreasing
function defined for real \( x \geq 0 \). For each such function \( \psi \) we define another
function by
\[
\tilde{\psi}(x) = \int_{1/2}^x \frac{\psi(t)}{t} dt, \quad x \geq \frac{1}{2}.
\]

Throughout this paper \( C \) denotes a positive constant, not necessarily the same at
each occurrence.

A function \( \varphi \) is almost increasing for \( x > 0 \) if there exists a positive constant \( c \)
such that \( x_1 < x_2 \) implies \( \varphi(x_1) \leq c\varphi(x_2) \). An almost decreasing function is defined
similarly.

Let \( m \) be a harmonic function in the open unit disc \( U \) and, as usual, denote
\[
M_p(r, u) = \left( \frac{1}{2\pi} \int_0^{2\pi} |u(re^{it})|^p \, dt \right)^{1/p}, \quad 0 < p < \infty,
\]
and
\[
M_\infty(r, u) = \sup \{|u(re^{it})|, 0 \leq t \leq 2\pi \}.
\]

Assuming that \( \psi(x)/x^a \) is almost decreasing for some \( a > 0 \), A. Shields and D.
Williams in [4] showed that if \( M_\infty(r, u) = O(\psi(1/(1-r))), \, r \to 1, \) then its con-
jugate \( v \) satisfies \( M_\infty(r, v) = O(\psi^p(1/(1-r))), \, r \to 1. \) They also showed that this
Theorem remains valid if we replace \( M_\infty(r, u) \) by \( M_1(r, u) \). If \( 1 < p < \infty \), the well-known theorem of M. Riesz [1, p. 54] says that \( M_p(r, u) = O(\psi(1/(1 - r))) \) implies \( M_p(r, v) = O(\psi(1/(1 - r))) \).

In this paper we shall be concerned only with means \( M_p(r, u) \) when \( 0 < p < 1 \). Hardy and Littlewood [3] proved that if \( M_p(r, u) = O(1) \) for some \( 0 < p < 1 \), then its conjugate \( v \) satisfies \( M_p(r, v) = O((\log 1/(1 - r))^{1/p}) \). They also showed that if \( M_p(r, u) = O((1 - r)^{-a}), a > 0, 0 < p < 1 \), then \( M_p(r, v) \) satisfies the same growth condition. We fill the gap between these two results.

**Theorem.** Let \( u \) be harmonic in the unit disc. If there exists \( a > 0 \) such that \( \psi(x)/x^a \) is almost decreasing for \( x \geq 1/2 \) and if \( M_p(r, u) = O(\psi(1/(1 - r))) \), for some \( p, 0 < p < 1 \), then the harmonic conjugate \( v \) satisfies

\[
M_p(r,v) = O(\left(\frac{\psi^p(1/(1 - r))}{1/(1 - r)}\right)^{1/p}).
\]

If \( \psi(x) \) grows like \( x^a, a > 0 \), then so does \( (\psi^p)^{1/p} \) and one obtains the theorem of Hardy and Littlewood. If \( \psi(x) = 1 \), then \( (\psi^p)^{1/p} \) grows like \( (\log x)^{1/p} \), thus we recapture the bounded case mentioned above.

**2. Proof of the theorem.** We will need a lemma.

**Lemma.** Let \( \psi \) satisfy the conditions of the theorem. If \( 0 < p < 1 \), then there exists \( C > 0 \) such that, for all \( x \geq 1 \),

\[
(\tilde{\psi}(x))^p \leq C\tilde{\psi}^p(x).
\]

**Proof.** Since \( \psi \) is nondecreasing we have

\[
\tilde{\psi}(x) = \int_{1/2}^{x} \frac{\psi(t)}{t} \, dt \leq (\psi(x))^{1-p}\tilde{\psi}^p(x).
\]

By Lemma 1(ii) of [4], \( \tilde{\psi} \) grows faster than \( \psi \); there exists \( C > 0 \) such that, for all \( x \geq 1 \), \( \psi(x) \leq C\tilde{\psi}(x) \). Hence,

\[
(\tilde{\psi}(x))^p \leq C\tilde{\psi}^p(x).
\]

**Proof of the theorem.** Without loss of generality we may suppose that \( u \) is real and \( u(0) = 0 \). Let \( f(z) = u(z) + iv(z) = \sum_{n=1}^{\infty} \hat{f}(n) z^n \) be a holomorphic function on \( U \). The fractional derivative of \( f \) of first order is defined as

\[
f^{[1]}(z) = \sum_{n=1}^{\infty} (n + 1) \hat{f}(n) z^n.
\]

Note that

\[
f(z) = \int_{0}^{1} f^{[1]}(tz) \, dt.
\]
Let \( r_n = 1 - 2^{-n} \). Then
\[
|f(re^{i\theta})|^p = |f(z)|^p = \left| \int_0^1 f^{[1]}(tz) \, dt \right|^p \\
\leq \left( \sum_{n=1}^{\infty} \int_{r_n}^{r_{n+1}} \sup_{0 \leq \rho \leq r_n} |f^{[1]}(tz)| \, d\rho \right)^p \\
\leq \left( \sum_{n=1}^{\infty} \sup_{0 \leq \rho \leq r_n} |f^{[1]}(tz)| 2^{-n} \right)^p \\
\leq \sum_{n=1}^{\infty} 2^{-np} \sup_{0 \leq \rho \leq r_n} |f^{[1]}(tz)|^p \\
\leq C \sum_{n=1}^{\infty} \sup_{0 \leq \rho \leq r_n} |f^{[1]}(tz)| \int_{r_n}^{r_{n+1}} (1 - \rho)^{p-1} \, d\rho \\
\leq C \sum_{n=1}^{\infty} \int_{r_n}^{r_{n+1}} \sup_{0 \leq \rho \leq r_n} |f^{[1]}(tz)|^p (1 - \rho)^{p-1} \, d\rho \\
\leq C \int_{0}^{1} (1 - \rho)^{p-1} \sup_{0 \leq \rho \leq r_n} |f^{[1]}(tz)|^p \, d\rho.
\]

If we now integrate on \( \theta \) and use the Hardy-Littlewood maximal theorem [1, p. 12] we obtain

(1) \[ M_p(r, f)^p \leq C \int_0^1 (1 - t)^{p-1} M_p(tr, f^{[1]})^p \, dt. \]

T. Flett [2, p. 762] proved that if \( 0 < p < 1 \) and \( 1/3 < r < 1 \), then
\[ M_p(r, f^*)^p \leq C (1 - r)^{-p-1} \int_{(3r-1)/2}^{(1+r)/2} M_p(t, u)^p \, dt. \]

Thus,

(2) \[ M_p(r, f^*)^p \leq C (1 - r)^{-p-1} \int_{(3r-1)/2}^{(1+r)/2} \psi^p \left( \frac{1}{1-t} \right) \, dt \\
= C (1 - r)^{-p-1} \int_{2(1-r)^{1/3}}^{2(1-r)^{-1}} \left[ \psi(t) \right]^p \cdot t^{-2} \, dt \\
= C (1 - r)^{-p-1} \int_{2(1-r)^{1/3}}^{2(1-r)^{-1}} \left[ \psi(t) / t^a \right]^p t^{a p - 2} \, dt \\
\leq C (1 - r)^{a p - p - 1} \int_{2(1-r)^{-1/3}}^{2(1-r)^{-1}} t^{a p - 2} \, dt \\
\leq C (1 - r)^{-p} \psi^p \left( \frac{1}{1-r} \right). \]

The inequality

(3) \[ M_p(r, f^{[1]}) \leq C M_p(r, f^*) \]

is obvious since \( f^{[1]}(z) = f(z) + zf'(z) \).
Combining (1), (2) and (3) we obtain

\[ M_p(r, f)^p \leq C \int_0^1 (1 - t)^{p-1}(1 - tr)^{-p} \psi_p \left( \frac{1}{1 - tr} \right) dt. \]

By Lemma 1(iii) of [4],

\[ \psi_p \left( \frac{1}{1 - r} \right) \leq C(1 - r) \sum_{n=0}^\infty \psi_p(n)r^n. \]

Hence, from (4), it follows that

\[ M_p(r, f)^p \leq C \int_0^1 \left( \frac{1 - rt}{1 - t} \right)^{1-p} \left( \sum_{n=0}^\infty \psi_p(n)r^n t^n \right) dt \]

\[ \leq C \left( \sum_{n=0}^\infty \psi_p(n)r^n \left( \int_0^r \left( \frac{1 - rt}{1 - t} \right)^{1-p} t^n dt \right) \right) \]

\[ + \sum_{n=0}^\infty \psi_p(n)r^n \left( \int_r^1 \left( \frac{1 - rt}{1 - t} \right)^{1-p} t^n dt \right) \].

We now show that each term in (5) is $O(\hat{\psi}_p(1/(1 - r)))$. Applying Lemma 1(v) of [4] to the function $\psi_p(x)$ we have

\[ \sum_{n=0}^\infty \psi_p(n) \frac{r^n}{n + 1} \leq C \hat{\psi}_p \left( \frac{1}{1 - r} \right). \]

Hence,

\[ \sum_{n=0}^\infty \psi_p(n)r^n \left( \int_0^r \left( \frac{1 - rt}{1 - t} \right)^{1-p} t^n dt \right) \leq C \sum_{n=0}^\infty \psi_p(n) \left( \int_0^1 t^n dt \right) r^n \]

\[ \leq C \sum_{n=0}^\infty \frac{\psi_p(n)}{n + 1} \leq C \hat{\psi}_p \left( \frac{1}{1 - r} \right). \]

It is a simple consequence of Jensen's inequality that

\[ (1 - r) \sum_{n=0}^\infty \frac{\psi_p(n)}{n + 1} r^n \leq \left( (1 - r) \cdot \sum_{n=0}^\infty \frac{\psi_p(n)}{n + 1} \right)^p. \]

For the second term, we have

\[ \sum_{n=0}^\infty \psi_p(n)r^n \left( \int_r^1 \left( \frac{1 - rt}{1 - t} \right)^{1-p} t^n dt \right) \]

\[ \leq C(1 - r)^{1-p} \sum_{n=0}^\infty \psi_p(n) \left( \int_0^1 (1 - t)^{p-1} t^n dt \right) r^n \]

\[ \leq C(1 - r)^{1-p} \sum_{n=0}^\infty \psi_p(n) \frac{\Gamma(p)\Gamma(n + 1)}{\Gamma(n + p + 1)} r^n \]

\[ \leq C(1 - r)^{1-p} \sum_{n=0}^\infty \frac{\psi_p(n)}{(n + 1)^p} r^n. \]
From (6), Lemma 1(v) of [4], and the lemma it follows that the second term on the right-hand side of (5) is also $O(\hat{\psi}^p(1/(1 - r)))$.

3. Remarks. A function $\psi$ is normal if there exist $a, b > 0$ such that $\psi(x)/x^a$ is almost decreasing for $x \geq 1/2$ and $\psi(x)/x^b$ is almost increasing for $x \geq 1/2$. If $0 < p < 1$ and $\psi$ is normal, then $\hat{\psi}$, $\hat{\psi}$ and $(\hat{\psi}^p)^{1/p}$ have the same rate of growth. Thus, when $\psi$ is normal the theorem says that if $u$ is harmonic in the unit disc and $M_p(r, u) = O(\psi(1/(1 - r)))$, then $M_p(r, v) = O(\psi(1/(1 - r)))$. If $\psi$ satisfies the conditions of the theorem, but is not normal, then $(\hat{\psi}^p)^{1/p}$ grows at a faster rate than $\hat{\psi}$. I do not know whether the theorem is best possible, i.e., is there a harmonic function $u$ such that

$$M_p(r, u) = O\left(\psi\left(\frac{1}{1 - r}\right)\right) \quad \text{and} \quad M_p(r, v) \geq C\left(\hat{\psi}^p\left(\frac{1}{1 - r}\right)\right)^{1/p}.$$

We note that Hardy and Littlewood proved in [3] that if $k$ is a positive integer, $p = 1/(k + 1)$ and $f(z) = \exp(\frac{1}{2} k \pi i)(1 - z)^{-k-1}$ then $M_p(r, \text{Re} f)$ is bounded and $M_p(r, f) \sim (\log(1/(1 - r)))^{1/p}$, $r \to 1$.

REFERENCES


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