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ORTHOGONALITY AND THE HAUSDORFF DIMENSION OF THE MAXIMAL MEASURE

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ABSTRACT. In this paper the orthogonality properties of iterated polynomials are shown to remain valid in some cases for rational maps. Using a functional equation fulfilled by the generating function, the author shows that the Hausdorff dimension of the maximal measure is a real analytical function of the coefficients of an Axiom A rational map satisfying the property that all poles of f and zeros of $f'(z)$ have multiplicity one.

Here we will consider f a rational map such that the Julia set (see [**1**]) is bounded and f is of the form $f(z) = P(z)(Q(z))^{-1}$, where $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, $Q(z) = b_d z^d + b_{d-1}z^{d-1} + \cdots + b_1 z + b_0$, where $a_i \in \mathbf{C}$, $b_j \in \mathbf{C}$, $b_d \neq 0$, $n > 2$, and $d < n$.

In [**6**, **8**, and **10**] it was shown that for f a rational map there exists just one f-invariant probability measure u such that, for any continuous function Φ,

$$\int \Phi(x)\, du(x) = n^{-1} \int \sum_{i=1} \Phi(z_i(x))\, du(x),$$

where $z_i(x)$, $i \in \{1, \ldots, n\}$, are the roots of $f(z) = x$, counted with multiplicity, and this is the measure of maximum entropy. This measure is called the maximal measure, and it has entropy $\log n$. For f such that $f(\infty) = \infty$ and $J(f)$ bounded, this measure is the equilibrium measure for the logarithm potential if and only if f is a polynomial [**1**, **9**].

Let $F(z)$ be the only one such that $F(z)/z$ is analytic near ∞, $F(z) \sim z$ as $z \to \infty$, and

$$F'(z)F(z)^{-1} = \int (z-x)^{-1} du(x) = z^{-1}\left(\sum_{m=0}^{\infty} M_m z^{-m}\right),$$

where $M_m = \int x^m du(x)$ for $m \in N$ (see [**2**]) are the m-moments of u.

Note that $M_0 = 1$, and the expansion is valid only when the Julia set is bounded, which implies either $d < n-1$ or $d = n-1$, and $|b_d| < 1$ or $|b_d| = 1$, and there is a Siegel disk around infinity.

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We will consider $d = n - 1$ in Theorems 1 and 2 just to simplify the formulas. The same result can be easily obtained in the same way in the general case $d < n$. In Theorems 3 and 4, the interesting case is for $d = n - 1$, and the formulas of Theorem 1 will be used there.

THEOREM 1. *Let $s_m = \sum_{i=1}^n p_i^m$ and $t_m = \sum_{j=1}^d q_j^m$, where $d = n - 1$ and p_i and q_j are respectively the zeros of P and Q. Let a_k^m be the coefficient of z^{-k} in the Laurent series in ∞ of $f(z)^{-m}$ where $m, k \in \mathbb{N}$, then M_m is obtained recursively by*

$$M_m = \left(n - a_m^m\right)^{-1}\left[s_m + \sum_{j=1}^{m-1} M_j \sum_{i=0}^{m-j} a_{m-i}^j (s_i - t_i)\right]. \tag{1}$$

PROOF. The following functional equation was obtained in [9]:

$$f'(z)\int (f(z) - x)^{-1}\,du(x) = n\int (z - x)^{-1}\,du(x) - \sum_{i=1}^{d}(z - q_i)^{-1}.$$

To obtain the Laurent series in ∞ of

$$f'(z)\int (f(z) - x)^{-1}\,du(x) = f'(z)f(z)^{-1}\sum_{m=0}^{\infty} M_m f(z)^{-m},$$

we have to obtain the Laurent series of $M_m f'(z)f(z)^{-(m+1)}$. This series is obtained in the following way:

$$\begin{aligned} M_m f'(z)f(z)^{-1}f(z)^{-m} &= M_m\left(P'(z)P(z)^{-1} - Q'(z)Q(z)^{-1}\right)f(z)^{-m} \\ &= M_m z^{-1}\left(\sum_{i=0}^{\infty}(s_i - t_i)z^{-i}\right)\left(\sum_{k=m}^{\infty} a_k^m z^{-k}\right) \\ &= M_m z^{-1}\sum_{j=0}^{\infty}\left(\sum_{i=0}^{j} a_{m+j-i}^m (s_i - t_i)\right)z^{-(m+j)}. \end{aligned}$$

We point out that $a_m^m = (b_d)^m$ for $m \geqslant 0$, the first term in the above expression is $M_m b_d^m(s_0 - t_0)z^{-(m+1)}$, and we have $(s_0 - t_0) = n - d = 1$.

The Laurent series in ∞ of $f'(z)\int(f(z) - x)^{-1}\,du(x)$ is

$$\begin{aligned} & f'(z)f(z)^{-1}\left(\sum_{m=0}^{\infty} M_m f(z)^{-m}\right) \\ &\quad = \sum_{m=0}^{\infty} M_m f'(z)f(z)^{-1}f(z)^{-m} \\ &\quad = z^{-1}\left(\sum_{v=0}^{\infty}\left[\left(\sum_{j=1}^{v} M_j \sum_{i=0}^{v-j} a_{v-i}^j (s_i - t_i)\right) + (s_v - t_v)\right]z^{-v}\right). \end{aligned}$$

The Laurent development of

$$n\int (z - x)^{-1}\,du(x) - \sum_{i=1}^{d}(z - q_i)^{-1}$$

is

$$z^{-1}\left(\sum_{v=0}^{\infty}(nM_v - t_v)z^{-v}\right).$$

Therefore

$$nM_m - t_m = (s_m - t_m) + M_m a_m^m + \sum_{j=1}^{m-1} M_j \left(\sum_{i=0}^{m-j} a_{m-i}^j (s_i - t_i) \right).$$

Finally, M_m can be obtained inductively by

$$M_m = (n - a_m^m)^{-1} \left[s_m + \sum_{j=1}^{m-1} M_j \left(\sum_{i=0}^{m-j} a_{m-i}^j (s_i - t_i) \right) \right].$$

DEFINITION 1. f is expanding if there exists a $k \in \mathbb{N}$ such that $|(f^k)'(x)| > 1$ for any z in the Julia set.

DEFINITION 2. The Hausdorff dimension of a measure u is the inf{Hausdorff dimension of Λ for all measurable sets such that $u(\Lambda) = 1$}.

Ruelle [12] showed that the Hausdorff dimension of the Julia set of an expanding rational map is a real analytic function of the coefficients. Here we will show

THEOREM 2. *Suppose f_λ is a family of expanding rational maps with coefficients depending analytically on $\lambda \in \mathbb{R}$ such that $f_\lambda(z)$ has all poles and $f_\lambda'(z)$ has all zeros with algebraic multiplicity one. Then Hausdorff dimension of the maximal measure of f_λ is real analytic with respect to the parameter λ. If all zeros and all poles are respectively in the same component of $\mathbb{C} - J(f_\lambda)$, then the condition on the zeros and poles is unnecessary.*

PROOF. By [11] the Hausdorff dimension of u satisfies

$$HD(u) = \text{entropy of } u \left(\int \log|f'(x)|\, du(x) \right)^{-1}$$

$$= \log n \left(\sum_{i=1}^{r} \int \log|x - r_i|\, du(x) - \sum_{j=1}^{v} \int \log|x - v_j|\, du(x) - \log b_{n-1} \right)^{-1},$$

where r_i and v_j are resepctively the zeros and poles of f' counted with multiplicity.

Since $\log|F(z)| = \int \log|z - x|\, du(x)$, we have

$$HD(u) = \log d \left(\sum_{i=1}^{r} F(r_i) - \sum_{j=1}^{v} F(v_j) - \log b_{n-1} \right)^{-1}.$$

We claim that the coefficients of the Laurent series of $F(z)$ depend analytically on the coefficients of $f(z)$. From [7, Theorem 17.3.2] the coefficients of $F(z)$ depend analytically on the moments M_m. Now, by (1), each moment M_m is a finite sum of s_j, t_j, a_j^k, which are themselves analytic on the coefficients of $f(z)$. Therefore the claim is proved.

Now since the sum of the values of an analytic map in the roots of a polynomial is an analytic function of the coefficients of the polynomial, we conclude that the Hausdorff dimension of the maximal measure is a real analytic function of the coefficients of $f(x)$.

Consider the sequence $\{f^n(z)\}$, $n \in \mathbb{N}$, where $f^0(z) = z$ and $f^n(z) = f \circ f^{n-1}(z)$. In [2] conditions were given for the orthogonality of the sequences $\{f^n\}$ with respect to the measure u when f is a polynomial (that is, $\int f^m(z) f^n(z)\, du(z) = 0$ for $m \neq n$). See also [3, 4 and 5].

Here we are using a nonhermitian scalar product similar to the one used in [2].

EXAMPLE. For $f(z) = z^n$ the maximal measure is Lebesgue measure on the unit circle, and orthogonality is a consequence of the orthogonality of the Fourier series.

For f a rational map such that $f(\infty) = \infty$, the interesting case is obtained when $d = n - 1$ by the following theorem.

THEOREM 3. *Let $f(z) = P(z)Q(z)^{-1}$, where $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$, $Q(z) = b_{n-1}z^{n-1} + \cdots + b_0$, $b_{n-1} \neq 0$, and the Julia set bounded. Then*

$$\int f^{m+1}(z) f^m(z)\, du(z) = n^{-1}(b_{n-1}M_2 + a_{n-1}M_1)$$

with

$$M_1 = -(n - b_{n-1})^{-1} a_{n-1},$$

$$M_2 = (n - b_{n-1}^2)^{-1}\{s_2 - a_{n-1}(n - b_{n-1})^{-1}(b_{n-2} - a_{n-1}b_{n-1} + (s_1 - t_1)b_{n-1})\}.$$

PROOF. By the f-invariance of u we have

$$\begin{aligned}\int f^{m+1}(z) f^m(z)\, du(z) &= \int f(z) z\, du(z)\\ &= n^{-1}\int z \sum_{i=1}^{n} z_i^1(z)\, du(z) = n^{-1}\int z(b_{n-1}z - a_{n-1})\, du(z)\\ &= n^{-1}(b_{n-1}M_2 - a_{n-1}M_1),\end{aligned}$$

and the theorem follows from (1).

REMARK 1. This theorem gives us necessary and sufficient conditions for $\int f^m(z) f^n(z)\, du(z) = 0$ for $m > n$ in terms of the coefficients of f^{m-n}, as explained by the next theorem.

THEOREM 4. *Let $f(z)$ be a rational map as above such that $a_{n-1} = a_{n-2} = 0$, $b_{n-1} \neq n$, $b_{n-1}^2 \neq n$. Then $\{f^n(z)\}$ satisfies $\int f^m(z) f^n(z)\, du(z) = 0$ for $m \neq n$.*

PROOF. Since $s_1 = -a_{n-1}$ and $s_2 = a_{n-1}^2 - 2a_{n-2}$, we have from (1) that $M_1 = 0$ and $M_2 = 0$. For $m > n$, f^{m-n} and f have the same maximal measure [6]. Therefore, using the same argument as for Theorem 3,

$$\begin{aligned}\int f^m(z) f^n(z)\, du(z) &= \int f^{m-n}(z) z\, du(z) = \int (cz + d) z\, du(z)\\ &= cM_2 + dM_1, \quad \text{where } c, d \in \mathbf{C}.\end{aligned}$$

Since $M_1 = M_2 = 0$, the proposition follows.

REMARK 2. If one considers the case of real rational maps such that the Julia set is contained on $\mathbb{R}$, one recovers orthogonality with respect to the usual inner product. Note that $b_{n-1} \neq n$ and $b_{n-1}^2 \neq n$ are automatically satisfied when $J(f)$ is bounded.

REFERENCES

1. H. Brolin, *Invariant sets under iteration of rational functions*, Ark. Mat. **6** (1966), 103–144.

2. M. F. Barnsley, J. S. Geronimo and A. N. Harrington, *Orthogonal polynomial associated with invariant measure on the Julia set*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), 381–384.

3. M. F. Barnsley and A. N. Harrington, *Moments of balanced measures on Julia sets*, Trans. Amer. Math. Soc. **284** (1984), 271–280.

4. D. Bessis, D. Mentha and P. Moussa, *Orthogonal polynomials on a family of Cantor sets and the problem of iterations of quadratic mapping*, Lett. Math. Phys. **6** (1982), 123–140.

5. D. Bessis and P. Moussa, *Orthogonality properties of iterated polynomial mapping*, Comm. Math. Phys. **88** (1983), 503–529.

6. A. Freire, A. Lopes and R. Mañé, *An invariant measure for rational maps*, Bol. Soc. Brasil. Mat. **14** (1983), 45–62.

7. E. Hille, *Analytic function theory*, Blaisdell, New York, 1963.

8. V. Lubitsh, *Entropy properties of rational endomorphisms of the Riemann sphere*, Ergodic Theory Dynamical Systems **3** (1983) 351–383.

9. A. Lopes, *Equilibrium measures for rational maps*, Ergodic Theory Dynamical Systems (to appear).

10. R. Mañé, *On the uniqueness of the maximizing measure for rational maps*, Bol. Soc. Brasil. Mat. **14** (1983), 27–43.

11. A. Manning, *The dimension of the maximal measure for a rational map*, Ann. of Math. (2) **3** (1984), 425–430.

12. D. Ruelle, *Repellers for real analytic maps*, Ergodic Theory Dynamical Systems **2** (1982), 99–107.

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