UNIFORM ERGODIC THEOREMS
FOR LOCALLY INTEGRABLE SEMIGROUPS
AND PSEUDO-RESOLVENTS

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Abstract. We study uniform ergodicity (at \( \infty \)) of a locally integrable operator semigroup \( T(\cdot) \) of type \( w_0 \) under a suitable condition which is weaker than the usual one \( "w_0 \leq 0" \). We also give a precise characterization of the uniform Cesàro-ergodicity for semigroups of class \((0,A)\). To prove the part of Abel-ergodicity we first prove a general uniform ergodic theorem for pseudo-resolvents.

1. Introduction. Let \( B(X) \) be the Banach algebra of all bounded linear operators on a Banach space \( X \), and let \( \{T(t); t > 0\} \) be a family in \( B(X) \) with the properties:
(i) \( T(s + t) = T(s)T(t) \) for all \( s, t > 0 \); (ii) for each \( x \in X \) the function \( T(\cdot)x \) is Bochner integrable with respect to the Lebesgue measure over every finite subinterval of \((0, \infty)\). Such a family \( T(\cdot) \) is called a locally integrable semigroup. It is known to be strongly continuous on \((0, \infty)\).

The type of \( T(\cdot) \) is the number \( w_0 := \inf_{t > 0} t^{-1} \log \|T(t)\| < \infty \). For every \( \lambda \) with \( \Re \lambda > w_0 \) and \( x \in X \) the Bochner integral
\[
R(\lambda)x := \int_0^\infty e^{-\lambda t}T(t)x \, dt
\]
exists and defines a bounded linear operator \( R(\lambda) \) on \( X \). The function \( R(\lambda) \), \( \Re \lambda > w_0 \) (called the Laplace transform of \( T(\cdot) \)), satisfies the first resolvent equation
\[
R(\lambda) - R(\mu) = (\lambda - \mu)R(\lambda)R(\mu),
\]
that is, \( R(\cdot) \) is a pseudo-resolvent on \( \{ \lambda \in \mathbb{C}; \Re \lambda > w_0 \} \) (cf. [2, p. 510]). \( R(\cdot) \) has a unique maximal extension satisfying the above equation, and the domain of definition \( \Omega \) of this maximally extended pseudo-resolvent, which we shall still denote by \( R(\cdot) \), is an open subset of the complex plane \( \mathbb{C} \), on which \( R(\cdot) \) is analytic and cannot be continued analytically beyond (cf. [2, pp. 188–189]). We denote \( w := \inf\{ u \in (-\infty, \infty); \lambda \in \Omega \text{ for all } \lambda \text{ with } \Re \lambda > u \} \). Then \( w \leq w_0 \).
One can also define the Laplace transform $R_s(\cdot)$ of $T(\cdot)$ in the following weaker sense:

$$R_s(\lambda) := \lim_{t \to -\infty} \int_0^t e^{-\lambda s} T(s) \, ds,$$

where the integral represents the operator which maps each $x \in X$ to the corresponding Bochner integral $\int_0^t e^{-\lambda s} T(s)x \, ds$. It can be proved (cf. [1, p. 248]) that if for a number $\lambda_0$, $R_s(\lambda_0)$ exists, then $R_s(\lambda)$ exists for all $\lambda$ with $\Re \lambda > \Re \lambda_0$. Let us denote

$$\sigma := \inf\{u \in (-\infty, \infty); R_s(\lambda) \text{ exists for all } \lambda \text{ with } \Re \lambda > u\},$$

$$\sigma_a := \inf\{u \in (-\infty, \infty); R_s(u) \text{ exists}\},$$

$$\sigma_w := \inf\{u \in (-\infty, \infty); R_s(u) \text{ is analytic for all } \lambda \text{ with } \Re \lambda > u\}.$$

It is clear that $\sigma \leq \sigma_a$, $\lambda \leq \sigma_a \leq \sigma_w$, and that $R_s(\lambda) = R(\lambda)$ for all $\lambda$ with $\Re \lambda > \sigma_w$, and hence, for all $\lambda$ with $\Re \lambda > \sigma_a$, by the uniqueness of analytic continuation.

It is known (cf. [1, Theorem 2.1]) that when $T(\cdot)$ is a semigroup of positive operators on an ordered Banach space, $R_s(\cdot)$ is analytic on $\{\lambda \in \mathbb{C}; \Re \lambda > \sigma\}$, so that $\lambda \leq \sigma_a = \sigma$. As will be seen in Proposition 7, this actually holds for all semigroups of class $(0, A)$.

By a semigroup of class $(0, A)$ we mean that $T(\cdot)$ is locally integrable and $\lambda R(\lambda)$ converges strongly to $I$ as $\lambda \to \infty$. In this case, the operator $A^0: x \to \lim_{t \to 0} t^{-1}(T(t) - I)x$ is densely defined and closable (cf. [2, pp. 342–344]). The closure $A$ of $A^0$ is called the infinitesimal generator of $T(\cdot)$. $T(\cdot)$ is of class $(C_0)$ if $T(t)$ converges strongly to $I$ as $t \to 0^+$. In this case, $T(\cdot)$ is also of class $(0, A)$ and $A^0$ is closed (see [2, p. 347]).

Let $S(t): x \to \int_0^t T(s) \, ds (x \in X)$. The operators $C(t) := t^{-1}S(t)$, $t > 0$, are the Cesàro averages of $T(\cdot)$, and the operators $A(\lambda) := \lambda R(\lambda)$, $\lambda > \sigma_a$, are the Abel averages of $T(\cdot)$. Uniform ergodic theorems are concerned with the uniform operator convergence of $C(t)$ as $t \to \infty$ and of $A(\lambda)$ as $\lambda \to 0^+$. The result of Hille and Phillips [2, Theorem 18.8.4] deals with the uniform Abel-ergodicity of semigroups of class $(A)$ (a class slightly larger than $(0, A)$) under the assumption “$\sigma_w \leq 0$”. The theorem of Lin [3] treats uniform Cesàro-ergodicity and Abel-ergodicity for semigroups of class $(C_0)$ under the assumption “$\lim_{t \to \infty} ||T(t)||/t = 0$”, which also implies $\sigma_w \leq 0$. However, as will be shown by an example in §3, it is possible for a semigroup to be uniformly ergodic while $-\infty = \lambda = \sigma = \sigma_a < 0 < \sigma_w$. Thus these well-known theorems do not apply universally.

The main purpose of this paper is to establish in §3 a uniform ergodic theorem (Theorem 4) for a general locally integrable semigroup, assuming the weaker condition “$\sigma_a \leq 0$”. Under this assumption, the condition “$\lim_{t \to \infty} ||T(t)||/t = 0$” (but not $\lim_{t \to \infty} ||T(t)||/t = 0$) is necessary for uniform Cesàro-ergodicity of $T(\cdot)$. This is unlike the discrete case, where $||T^n||/n \to 0$ is necessary for the uniform ergodicity of $\{T^n\}$. When $T(\cdot)$ is a semigroup of class $(0, A)$, the uniform
Cesàro-ergodicity can be characterized precisely (Theorem 6). We shall begin with a uniform ergodic theorem (Theorem 1) for a general pseudo-resolvent. It will apply in §3 to provide conditions for uniform Abel-ergodicity of $T(\cdot)$.

2. Uniform ergodic theorems for pseudo-resolvents. In this section $R(\cdot)$ is a general pseudo-resolvent on an open subset $\Omega$ of the complex plane $\mathbb{C}$.

**Theorem 1.** Suppose that $0 \in \overline{\Omega}$. Then the following two statements are equivalent:

1. There is a $P \in B(X)$ such that $\|\lambda R(\lambda) - P\| \to 0$ as $\lambda \to 0$, $\lambda \in \Omega$.
2. $\|\lambda^2 R(\lambda)\| \to 0$ as $\lambda \to 0$, and the range $R(\lambda R(\lambda) - I)$ of $\lambda R(\lambda) - I$ is closed for some (and hence all) $\lambda \in \Omega$.

**Proof.** Note first that the range $R(\lambda R(\lambda) - I)$ and the null space $N(\lambda R(\lambda) - I)$ of $\lambda R(\lambda) - I$ are independent of $\lambda$ (see [6, p. 215]).

$(1) \Rightarrow (2)$. It is known from the mean ergodic theorem [6, p. 217] that $P$ is the projection onto $N(\lambda R(\lambda) - I)$ along $R(\lambda R(\lambda) - I)$. The fact that $\|\lambda R(\lambda) N(P)\| = \|(\lambda R(\lambda) - P) N(P)\| \leq \|\lambda R(\lambda) - P\| \to 0$ as $\lambda \to 0$ implies that $(\lambda R(\lambda) - I) N(P) = R(\lambda R(\lambda) - I)$, i.e., $R(\lambda R(\lambda) - I)$ is closed.

$(2) \Rightarrow (1)$. Fix a $\mu \neq 0$. Since $\mu R(\mu) - I$ has closed range, there exists a $M > 0$ such that each $y$ in $R(\mu R(\mu) - I)$ can be written as $y = (\mu R(\mu) - I)x$ for some $x$ satisfying $\|x\| \leq M\|y\|$. Using the resolvent equation we have

$$\|\lambda R(\lambda) y\| = \|\lambda R(\lambda) (\mu R(\mu) - I)x\| = \|((\mu - \lambda)^{-1}[\lambda^2 R(\lambda) - \lambda \mu R(\mu)]x\| \leq |\mu - \lambda|^{-1} \left[\|\lambda^2 R(\lambda)\| + |\lambda| \|\mu R(\mu)\|\right] M\|y\|,$$

which implies that $\|\lambda R(\lambda) R(\lambda R(\lambda) - I)\| \to 0$ as $\lambda \to 0$. Hence for small $\lambda$ the operator $K := (\lambda R(\lambda) - I) R(\lambda R(\lambda) - I)$ is invertible and so we have that $R(\lambda R(\lambda) - I) = R(K) = R((\lambda R(\lambda) - I)^2)$. From this one easily deduces that $X = N(\lambda R(\lambda) - I) + R(\lambda R(\lambda) - I)$.

To show that the summation is direct, let $y$ be in $N(\lambda R(\lambda) - I) \cap R(\lambda R(\lambda) - I)$. Then $y = \lambda R(\lambda) y$ for all $\lambda \in \Omega$, and, as shown above, $\lambda R(\lambda) y \to 0$ as $\lambda \to 0$. Hence $y = 0$ and so $X = N(\lambda R(\lambda) - I) \oplus R(\lambda R(\lambda) - I)$. Let $P$ be the projection onto $N(\lambda R(\lambda) - I)$ along $R(\lambda R(\lambda) - I)$. Clearly we have $\|\lambda R(\lambda) - P\| = \|0 \oplus [\lambda R(\lambda) R(\lambda R(\lambda) - I)]\| \to 0$ as $\lambda \to 0$.

**Corollary 2.** Suppose that the domain $\Omega$ of $R(\cdot)$ is unbounded. Then the following statements are equivalent:

1. There is a $Q \in B(X)$ such that $\|\lambda R(\lambda) - Q\| \to 0$ as $|\lambda| \to \infty$, $\lambda \in \Omega$.
2. $\|R(\lambda)\| \to 0$ as $|\lambda| \to \infty$, and $R(R(\lambda))$ is closed for some (and hence all) $\lambda \in \Omega$.
3. $R(\lambda) = Q(\lambda I - A)^{-1}$ for some $Q$, $A \in B(X)$ satisfying $Q^2 = Q$, $AQ = QA = A$.

**Proof.** Define $R_1(\lambda) := 1/\lambda - (1/\lambda^2) R(1/\lambda)$ for $\lambda \in \Omega_1 := \{\lambda \in \mathbb{C}; 1/\lambda \in \Omega\}$. An easy computation shows that $R_1(\cdot)$ is a pseudo-resolvent on $\Omega_1$, which has the limit point 0. Therefore, Theorem 1 applies to $R_1(\cdot)$, and the equivalence of (1) and
(2) follows by the facts that \(\|\lambda^2 R_1(\lambda)\| \to 0\) as \(\lambda \to 0\), \(\lambda \in \Omega_1\), if and only if \(\|R(\lambda)\| \to 0\) as \(|\lambda| \to \infty\), \(\lambda \in \Omega\), and that \(\|\lambda R_1(\lambda) - P\| \to 0\) as \(\lambda \to 0\) if and only if \(\|\lambda R(\lambda) - Q\| \to 0\) as \(|\lambda| \to \infty\), where \(Q = I - P\). "(3) \(\Rightarrow\) (1)" is obvious, and "(1) \(\Rightarrow\) (3)" is proved in Theorem 18.8.2 of [2].

**Remark.** The operators \(P\) and \(Q\) turn out to be the linear projections with \(R(P) = N(\lambda R(\lambda) - I), N(P) = R(\lambda R(\lambda) - I), R(Q) = R(R(\lambda))\) and \(N(Q) = N(R(\lambda))\) for all \(\lambda \in \Omega\).

3. Uniform ergodic theorems for locally integrable semigroups. All well-known uniform ergodic theorems for semigroups have been formulated for those of type \(w_0 < 0\). We shall first give an example of a uniformly ergodic semigroup of class \((C_0)\) which satisfies \(-\infty = \sigma_a < 0 < w_0\), and then prove a uniform ergodic theorem for general locally integrable semigroups under the assumption \(\sigma_a \leq 0\).

**Example.** Let \(1 < p < q < \infty\), and let \(X\) be the set of all Lebesgue measurable functions \(f\) on \((0, \infty)\) such that
\[
norm{f} := \left(\int_0^\infty e^{ps^2} |f(s)|^p ds\right)^{1/p} + \left(\int_0^\infty |f(s)|^q ds\right)^{1/q} < \infty.
\]
Then \((X, \| \cdot \|)\) is a Banach lattice which is reflexive whenever \(p > 1\). For \(\alpha \geq 0\) let \(T_\alpha(\cdot)\) be the semigroup defined by \((T_\alpha(t)f)(s) := e^{\alpha f(t + s)} (f \in X, s, t \geq 0)\). Then \(T_\alpha(t) = e^{\alpha t} T_0(t)\).

It was shown in [1] that \(\|T_0(t)\| = 1\) for all \(t \geq 0\) and for \(T_0(\cdot)\) \(w = \sigma_a = \sigma = -\infty\). Hence \(T_\alpha(\cdot)\) has type \(w_0 = \alpha\) and \(w = \sigma_a = \sigma = -\infty\) for all \(\alpha \geq 0\). Thus the infinitesimal generator \(A_\alpha = d/ds + \alpha I\) of \(T_\alpha(\cdot)\) has empty spectrum, and so
\[
\|A(\lambda)\| = \|\lambda(\lambda I - A_\alpha)^{-1}\| \to 0 \cdot \|A^{-1}_\alpha\| = 0 \quad \text{as } \lambda \to 0.
\]
Since \(T_\alpha(t)\) are positive operators, we have for any nonnegative function \(f\) in \(X\)
\[
C(t)f = t^{-1}\int_0^t T_\alpha(s)f ds \leq t^{-1}\int_0^t e^{s-t/\alpha} T_\alpha(s)f ds
\]
\[
\leq t^{-1}e^{t/\alpha}\int_0^\infty e^{-s/\alpha} T_\alpha(s)f ds = e^{t/\alpha} R(t^{-1})f = eA(t^{-1})f.
\]
It follows that \(\|C(t)\| \leq \|eA(t^{-1})\| \to 0\) as \(t \to \infty\) (cf. [4, pp. 81 and 230]). Hence \(T_\alpha(\cdot)\) is uniformly Abel and Cesàro ergodic to 0.

We shall need the following lemma in the proof of Theorem 4:

**Lemma 3.** Let \(T(\cdot)\) be a locally integrable semigroup, and let \(S(t)\) and \(R_\alpha(\lambda)\) be as defined in §1. The equality \((T(t) - I)R_\alpha(\lambda) = S(t)(\lambda R_\alpha(\lambda) - I)\) holds for all \(t > 0\) and \(\Re \lambda > \sigma\).

**Proof.** Integration by parts gives that
\[
R_\alpha(\lambda)x = \int_0^\infty e^{-\lambda u}T(u)x du = \lambda \int_0^\infty e^{-\lambda u}S(u)x du.
\]
Then we use the identity \((T(t) - I)S(u) = S(t)(T(u) - I)\) (see [5]) to obtain that
\[
(T(t) - I)R_\alpha(\lambda)x = \lambda \int_0^\infty e^{-\lambda u}(T(t) - I)S(u)x du
\]
\[
= \lambda \int_0^\infty e^{-\lambda u}S(t)(T(u) - I)x du = S(t)(\lambda R_\alpha(\lambda) - I)x.
\]
Theorem 4. Let $T(\cdot)$ be a locally integrable semigroup. Assume that $\sigma < 0$. Then the following statements are equivalent:

1. $T(\cdot)$ is uniformly Cesàro-ergodic.
2. $\|T(t)R(1)\|/t \to 0$ as $t \to \infty$, and $T(\cdot)$ is uniformly Abel-ergodic.
3. $\|T(t)R(1)\|/t \to 0$ as $t \to \infty$, and $\mathbb{R}(R(1) - I)$ is closed.

Proof. (1) $\Rightarrow$ (2). We have for each $x \in X$ and $a > 0$,

$$\|(A(\lambda) - P)x\| = \left\| \lambda^2 \int_0^\infty e^{-\lambda t}(S(t) - tP)x\,dt \right\|$$

$$\leq \left[ \lambda^2 \int_0^a e^{-\lambda t}(\|S(t)\| + \|tP\|)\,dt + \lambda^2 \int_a^\infty e^{-\lambda t}\|C(t) - P\|\,dt \right]\|x\|$$

$$\leq \left( \sup_{0 \leq t \leq a} \|S(t)\| + a\|P\| \right)\lambda^2 a + \sup_{t > a} \|C(t) - P\|\|x\|.$$ 

If $\|C(t) - P\| \to 0$ as $t \to \infty$, then it is easy to see from the above estimate that $\|A(\lambda) - P\| \to 0$ as $\lambda \to 0^+$. Then the fact that $\mathcal{N}(P) = \mathbb{R}(R(1) - I)$ (see the remark in §2) and Lemma 3 imply

$$\|(T(t) - I)R(1)\|/t = \|C(t)(R(1) - I)\| = \|(C(t) - P)(R(1) - I)\|$$

$$\leq \|(C(t) - P)\|\|R(1) - I\| \to 0 \quad (t \to \infty).$$

Hence the statement (2) holds when (1) holds.

"(2) $\Rightarrow$ (3)" is contained in Theorem 1.

(3) $\Rightarrow$ (1). First we prove that $\lim_{t \to \infty}\|T(t)R(1)\|/t = 0$ implies $\lim_{\lambda \to 0^+}\|\lambda^2 R(\lambda)\| = 0$. Given $\varepsilon > 0$, let $a > 0$ be such that $\|T(t)R(1)\| < \varepsilon t$ for all $t > a$. Using the resolvent equation we have for every $x \in X$

$$\|\lambda^2 R(\lambda)x\| = \left\| \lambda^2 \left[ R(1) + (1 - \lambda)(\lambda R)R(1) \right] x \right\|$$

$$\leq \lambda^2 \|R(1)\|\|x\| + |1 - \lambda|\lambda^2 \int_0^\infty e^{-\lambda t}\|T(t)R(1)x\|\,dt$$

$$\leq \lambda^2 \|R(1)\|\|x\| + |1 - \lambda| \left[ \lambda^2 \int_0^a \|T(t)R(1)x\|\,dt + \varepsilon \lambda^2 \int_a^\infty e^{-\lambda t}\,dt \,\|x\| \right]$$

$$\leq \left\{ \lambda^2 \|R(1)\| + |1 - \lambda|\left[ \lambda^2 \|W(a)\|\|R(1)\| + \varepsilon \right] \right\}\|x\|,$$

where $W(a)$ denotes the operator from $X$ to $L_1(X,[0,a])$ defined by $W(a)x = T(\cdot)x$, which is known to be bounded (cf. [2, p. 58]). It is easily seen from the above estimate that $\|\lambda^2 R(\lambda)\| \to 0$ as $\lambda \to 0^+$.

Now the statement (2) of Theorem 1 holds, and it was proved there that

$$X = \mathcal{N}(R(1) - I) \oplus \mathbb{R}(R(1) - I).$$

Let $K$ be the restriction of $R(1) - I$ to $\mathbb{R}(R(1) - I)$. Then $K$ is one-to-one, onto, and hence invertible. For $x \in \mathbb{R}(R(1) - I)$ let $y = K^{-1}x$. By Lemma 3 we have

$$\|C(t)x\| = \|C(t)(R(1) - I)y\| = \|t^{-1}(T(t) - I)R(1)K^{-1}x\|$$

$$\leq t^{-1}(\|T(t)R(1)\| + \|R(1)\|)\|K^{-1}\|\|x\|.$$
This shows that \(|C(t)|N(R(1) - I)| \to 0 as \(t \to \infty\). Let \(P\) be the projection onto \(R(R(1) - I)\) along \(R(R(1) - I)\). In order to prove that \(|C(t) - P|\) tends to 0 as \(t \to \infty\), it remains to show that the restriction of \(C(t)\) to \(N(R(1) - I)\) is an identity map for all \(t > 0\). It suffices to show that \(N(R(1) - I) \subset N(T(t) - I)\). Let \(x \in N(R(1) - I)\). Then \(x \in N(\lambda R(\lambda) - I)\) for all \(\lambda > \sigma_a\) [6, p. 215], so that
\[
(T(t) - I)x = \lambda(T(t) - I)R(\lambda)x
= \lambda S(t)(\lambda R(\lambda) - I)x = S(t)(\lambda^2 R(\lambda) - \lambda)x,
\]
which converges to 0 as \(\lambda \to 0^+\). Hence \(x\) belongs to \(N(T(t) - I)\). The proof is now completed.

**Corollary 5.** Let \(T(\cdot)\) be a locally integrable semigroup satisfying \(\sigma_a \leq 0\) and \(|T(t)R(1)|/t \to 0 (t \to \infty)\). Then the following statements are equivalent:

1. \(T(\cdot)\) is uniformly Cesàro-ergodic.
2. \(T(\cdot)\) is uniformly Abel-ergodic.
3. \(R(R(1) - I)\) is closed.

**Remarks.**

1. If \(T(\cdot)\) is of class \((0, A)\) with generator \(A\), one has that \(R(R(1) - I) = R((I - A)^{-1} - I) = R(A(I - A)^{-1}) = R(A)\). Thus the theorem of Lin [3] is a specialization of Corollary 5.

2. It follows from Theorem 4 that the semigroup \(T_a(\cdot)\) in the previous example satisfies \(|T_a(t)R(1)|/t \to 0 (t \to \infty)\), while \(|T_a(t)|/t = e^{\alpha t} \to \infty\) in case \(\alpha > 0\). Therefore, the hypothesis in Corollary 5 is in general strictly weaker than Lin's \((|T(t)|/t \to 0)\), and it cannot be further weakened.

The next theorem gives a precise characterization for the uniform Cesàro-ergodicity of \((0, A)\) semigroups.

**Theorem 6.** Let \(T(\cdot)\) be a semigroup of class \((0, A)\). Then \(T(\cdot)\) is uniformly Cesàro ergodic if and only if (i) \(\sigma \leq 0\), (ii) \(R(A)\) is closed, and (iii) \(|T(t)R(1)|/t \to 0 as t \to \infty\).

This theorem is deduced from Theorem 4 and the following two propositions.

**Proposition 7.** If \(T(\cdot)\) is a semigroup of class \((0, A)\), then \(\sigma = \sigma_a\).

**Proof.** Since \(\sigma \leq \sigma_a\), it suffices to show that if \(\Re \lambda > \sigma\), then \(\lambda I - A\) is invertible and \(R_s(\lambda) = (\lambda I - A)^{-1}\).

We first prove that \((\lambda I - A^0)R_s(\lambda)x = x\) for \(x \in D(A^0)\). Let
\[
A_h = h^{-1}(T(h) - I).
\]
Then we have
\[
A_h R_s(\lambda)x = A_h \lim_{t \to \infty} \int_0^t e^{-\lambda s} T(s) x \, ds
= h^{-1} \lim_{t \to \infty} \int_0^t e^{-\lambda s} [T(s + h)x - T(s)x] \, ds
= h^{-1}(e^{\lambda h} - 1) R_s(\lambda)x - h^{-1} \int_0^h e^{\lambda(s+h)} T(s) x \, ds,
\]
which tends to $\lambda R_s(\lambda)x - x$ as $h \to 0$ if $x \in D(A^0)$, by the continuity of $T(t)x$ at $t = 0$. Hence $R_s(\lambda)x \in D(A^0)$ and $A^0R_s(\lambda)x = \lambda R_s(\lambda)x - x$.

If $y \in X$, we can obtain a sequence $y_n \in D(A^0)$ such that $y_n \to y$. Then we have that $R_s(\lambda)y_n \to R_s(\lambda)y$ and $A^0R_s(\lambda)y_n - y_n = \lambda R_s(\lambda)y_n - y_n \to \lambda R_s(\lambda)y - y$.

Since $A$ is the closure of $A^0$, we have proved that $R_s(\lambda)X \subseteq D(A)$ and $(\lambda I - A)R_s(\lambda) = I$.

It remains to show that $AR_s(\lambda)x = R_s(\lambda)Ax$ for $x \in D(A)$. Given $x \in D(A)$ there exists a sequence $x_n \in D(A^0)$ such that $x_n \to x$ and $A^0x_n \to Ax$. Since $A^0R_s(\lambda)x_n = R_s(\lambda)A^0x_n$, by letting $h \to 0$ we obtain that $A^0R_s(\lambda)x_n \to R_s(\lambda)Ax$ as $n \to \infty$. This and the fact that $R_s(\lambda)x_n \to R_s(\lambda)x$ show that $R_s(\lambda)x \in D(A)$ and $AR_s(\lambda)x = R_s(\lambda)Ax$.

**Proposition 8.** If a locally integrable semigroup $T(\cdot)$ is strongly Cesàro-ergodic, then $\sigma \leq 0$, i.e. the Laplace transform $R_s(\lambda)$ exists for all $\lambda$ with $\Re \lambda > 0$.

**Proof.** The uniform boundedness principle implies that $\|C(t)\| \leq M$ for all $t > 1$. Let $\Re \lambda > 0$. We have for all $v > u > 0$ and $x \in X$

$$\left\| \int_u^v e^{-\lambda t}T(t)x \, dt \right\| \leq \left\| e^{-\lambda u}S(u)x \right\| v + \lambda \int_u^v e^{-\lambda t}S(t)x \, dt \right\| \leq \left\{ \left| e^{-\lambda u} \right| v + \left| e^{-\lambda u} \right| u + \left| \lambda \right| \int_u^v e^{-t \Re \lambda t} \, dt \right\} M \|x\|,$$

which shows that $\int_u^v e^{-\lambda t}T(t) \, dt \to 0$ as $u \to \infty$. Hence

$$R_s(\lambda) = \lim_{t \to \infty} \int_0^t e^{-\lambda t}T(t) \, dt$$

exists.

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**References**


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