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ALGEBRA DIRECT SUM DECOMPOSITION OF $C_R(X)$

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ABSTRACT. Let A and B be closed subalgebras of $C_R(X)$ with $1 \in A$ and $1 \notin B$. We give necessary and sufficient conditions for $A \oplus B = C_R(X)$.

1. Introduction, notations, and preliminaries. Throughout this paper X denotes a compact Hausdorff space, $C_R(X)$ denotes the Banach algebra of real-valued continuous functions on X with supremum norm, and A and B denote closed subalgebras of $C_R(X)$ with $1 \in A$ and $1 \notin B$. Fisher [2] has discussed some consequences of $A \oplus B = C_R(X)$. Also the necessary and sufficient conditions are known for $A \oplus B = C_R(X)$ when B is a closed ideal [1, Corollary 4.10]. In this paper we generalise the latter result for a closed algebra B using the ideas of Fisher and give examples to show that our conditions are independent.

If x, $y \in X$, we say that x is A-related to y, and write $x \overset{A}{-} y$, if $f(x) = f(y)$ for every f in A. This is an equivalence relation on X. Since each function in A is constant on each equivalence class, we call these equivalence classes the sets of constancy of A. We shall denote the corresponding quotient space by Q_A and the quotient map of X onto Q_A by π_A. Q_B and π_B are defined similarly.

Given x, $y \in X$, we say that there is an (A, B) chain from x to y, if there is an ordered set of points $\{x_0, x_1, \ldots, x_n, x_{n+1}\}$ with $x_0 = x$, $x_{n+1} = y$ such that either $x_i \overset{A}{-} x_{i+1} \overset{B}{-} x_{i+2}$ or $x_i \overset{B}{-} x_{i+1} \overset{A}{-} x_{i+2}$ $(i = 0, 1, 2, \ldots, n-1)$. We also say that the chain $\{x_0, x_1, \ldots, x_{n+1}\}$ has length $n+1$ provided $x_i \neq x_{i+1}$ $(i = 0, 1, 2, \ldots, n)$. The length of a chain C will be denoted by $l(C)$. We call x to be chain related to y if there exists an (A, B) chain from x to y. Again this is an equivalence relation and the corresponding equivalence classes are called (A, B) orbits. A chain of the form $\{x, x_1, x_2, \ldots, x_n, x\}$ where $x \neq x_1, \ldots, x_i \neq x_{i+1}, \ldots, x_n \neq x$ is called an (A, B) round trip [3, p. 205]. When the context is clear, we shall simply write chain, orbit, and round trip instead of (A, B) chain, (A, B) orbit, and (A, B) round trip.

We shall require a few elementary lemmas.

LEMMA 1.1. *let Z denote the set of common zeros of functions of B. Then Z is nonempty.* (*Recall that B is a closed subalgebra with* $1 \notin B$.)

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PROOF. For $g \in B$, define $\tilde{g}$ on Q_B by $\tilde{g}(\pi_B(x)) = g(x)$ for $x \in X$. Then $\tilde{g}$ is well defined and continuous on Q_B. If $\tilde{B} = \{\tilde{g}\colon g \in B\}$, then $\tilde{B}$ is a closed subalgebra of $C_R(Q_B)$ which separates points of Q_B. Since $1 \notin B$, $\tilde{B} \neq C_R(Q_B)$ and hence by the Stone-Weierstrass theorem, there is $t \in Q_B$ such that $\tilde{g}(t) = 0$ for every $\tilde{g} \in \tilde{B}$. Also such a t is unique. Hence $Z = \pi_B^{-1}(t) \neq \varnothing$.

In the sequel Z will denote the set of common zeros of elements of B. For $x \in X$, we shall say that there is a chain from x to Z if there exists a chain from x to some point in Z.

The following two lemmas are easy to prove.

LEMMA 1.2. *If there exists a chain from each point of X to Z, then the only orbit in X is X itself.*

LEMMA 1.3. *Suppose that there exists a chain from each point of X to Z. Then the following are equivalent*:

(i) *There is no round trip in X.*

(ii) *There exist a unique chain from each point of X to Z.*

2. Conditions for $A \oplus B = C_R(X)$.

THEOREM 2.1. *The following conditions are necessary and sufficient in order that*

$$A \oplus B = C_R(X). \tag{1}$$

(a_1) *From each point of X there exists a chain to Z.*

(a_2) *The chain in* (a_1) *is unique.*

(a_3) *The lengths of chains C_x (from x to Z) are uniformly bounded, i.e., there exists a positive integer m such that $l(C_x) \leqslant m$ for each $x \in X$.*

PROOF. Suppose that $A \oplus B = C_R(X)$. If P_A is the projection of $C_R(X)$ onto A, Fisher [2] has shown that there exists a chain from each point x of X to Z of length $\leqslant \|P_A\|$. Also by Lemma 1.2, Proposition 2 of [3] and Lemma 1.3, the chain from x to Z is unique. This proves the necessity part.

Conversely suppose that the conditions (a_1)–(a_3) are satisfied. By (a_1), Lemma 1.2, (a_2) and Proposition 2 of [3], $A + B$ is dense in $C_R(X)$. (In [3] it is assumed that $1 \in B$. However Proposition 2 of [3] holds even when $1 \notin B$, as is easy to see.) The condition (a_1) implies that $A \cap B = \{0\}$ and hence $A + B = A \oplus B$.

Next let $h \in C_R(X)$. Define $\Phi_h\colon X \to \mathbf{R}$ as follows:

If $x \in X$ has the chain to $p \in Z$ given by

$$x \overset{A}{-} x_1 \overset{B}{-} x_2 \overset{A}{-} x_3 - \cdots \overset{B}{-} x_{2n} \overset{A}{-} p, \tag{2}$$

$$\Phi_h(x) = h(x_1) - h(x_2) + h(x_3) - \cdots + h(x_{2n-1}) - h(x_{2n}) + h(p). \tag{3}$$

If the chain from x to p is given by

$$x \overset{B}{-} x_1 \overset{A}{-} x_2 - \cdots \overset{B}{-} x_{2n+1} \overset{A}{-} p, \tag{4}$$

$$\Phi_h(x) = h(x) - h(x_1) + h(x_2) - \cdots + h(x_{2n}) - h(x_{2n+1}) + h(p). \tag{5}$$

If $h \in A \oplus B$, then it is easy to see that $\Phi_h = P_A(h)$, P_A being the projection of $A \oplus B$ onto A. Let now $h \in C_R(X)$. Since $A \oplus B$ is dense in $C_R(X)$, there exists a sequence $\{h_n\}$ in $A \oplus B$ such that $h_n \to h$ uniformly on X. We shall show that $P_A(h_n) \to \Phi_h$ uniformly on X. Let $\varepsilon > 0$ be given. Then there exists n_0 such that

$$n \geqslant n_0 \Rightarrow |h_n(y) - h(y)| < \varepsilon/m \quad \text{for every } y \in X, \tag{6}$$

where m is as in (a_3). If now $x \in X$ has the chain to Z given by (2), we have for $n \geqslant n_0$, by (3) and (6),

$$\begin{aligned} |\Phi_h(x) - P_A(h_n)(x)| &= |\Phi_h(x) - \Phi_{h_n}(x)| \\ &\leqslant \sum_{i=1}^{2n} |h(x_i) - h_n(x_i)| + |h(p) - h_n(p)| \\ &< (2n+1)\varepsilon/m. \end{aligned}$$

Hence,

$$|\Phi_h(x) - P_A(h_n)(x)| < \varepsilon, \tag{7}$$

since $l(C_x) \leqslant m$. Similarly it can be shown that if C_x is of the form (4), then also (7) holds. Thus $P_A(h_n) \to \Phi_h$ uniformly on X. Since A is closed, $\Phi_h \in A$. Let $\Psi_h = h - \Phi_h$. Then

$$\Psi_h = \lim_{n\to\infty} (h_n - P_A(h_n)) = \lim_{n\to\infty} P_B(h_n),$$

where P_B is the projection of $A \oplus B$ onto B. Hence $\Psi_h \in B$ as B is closed. Thus $h = \Phi_h + \Psi_h$ with $\Phi_h \in A$ and $\Psi_h \in B$ for every $h \in C_R(X)$. Hence (1) holds and the proof is complete.

COROLLARY 2.2. *The following conditions are necessary and sufficient in order that* $A \oplus B = C_R(X)$.

(b_1) *Each set of constancy of A intersects each set of constancy of B in at most one point.*

(b_2) $A + B$ *is an algebra.*

(b_3) *From each point of X there is a chain to Z of length* $\leqslant m$, *where m is a fixed positive integer.*

PROOF. The condition (b_1) is equivalent to saying that $A + B$ separates points of X. Hence if $A \oplus B = C_R(X)$, (b_1) and (b_2) are satisfied. Also (b_3) is satisfied by Theorem 2.1.

Conversely if (b_1) and (b_2) are satisfied, then $A + B$ is dense in $C_R(X)$ by the Stone-Weierstrass theorem. Hence if a chain exists from $x \in X$ to Z, then it must be unique. Condition (b_3) with Theorem 2.1 now gives the result.

COROLLARY 2.3. *Suppose that A or B has only a finite number of nontrivial sets of constancy. If from each point in X there is a unqiue chain to Z, then* $A \oplus B = C_R(X)$.

PROOF. We note that every intermediate point of a chain from a point of X to Z has to belong to a nontrivial set of constancy of A or of B. Hence if A (or B) has only n nontrivial sets of constancy, then the length of each chain to $Z \leqslant 2n + 1$. Hence the conditions of Theorem 2.1 are satisfied and we have $A \oplus B = C_R(X)$.

Examples can be given to show that the condition of finite number of nontrivial sets of constancy of A or B is not necessary for $A \oplus B = C_R(X)$.

We give below examples to show that the condition of uniqueness of chains and uniform boundedness of their lengths in Theorem 2.1 are independent.

EXAMPLE 2.4. Let $X = \{1/n\colon n = 1, 2, \dots\} \cup \{0\}$ with the relative topology from $\mathbf{R}$. Let D_1 and D_2 be partitions of X given by

$$D_1 = \left\{\left\{0, \tfrac{1}{2}, 1\right\}, \left\{\tfrac{1}{4}, \tfrac{1}{3}\right\}, \left\{\tfrac{1}{6}, \tfrac{1}{5}\right\}, \dots, \left\{\tfrac{1}{2n}, \tfrac{1}{2n-1}\right\}, \dots\right\},$$

$$D_2 = \left\{\{1\}, \left\{\tfrac{1}{3}, \tfrac{1}{2}\right\}, \left\{\tfrac{1}{5}, \tfrac{1}{4}\right\}, \dots, \left\{\tfrac{1}{2n+1}, \tfrac{1}{2n}\right\}, \dots, \{0\}\right\}.$$

Then D_1 and D_2 are upper semicontinuous [4, Definition 5.2.3]. Let

$$A = \{f \in C_R(X)\colon f \text{ is constant on each member of } D_1\},$$

$$B = \{g \in C_R(X)\colon g \text{ is constant on each member of } D_2,\ g(1) = 0\}.$$

By upper semicontinuity of D_1 and D_2, the sets of constancy of A and B are precisely the members of D_1 and D_2 respectively [4, Exercise 7.5.7F(b)]. For $x = 1/k$, there is a chain $\{\frac{1}{k}, \frac{1}{k-1}, \dots, 1\}$ from x to Z and this is the only chain from x to Z. However, the lengths of these chains are not uniformly bounded. The function $h(x) = x$ $(x \in X)$ is in $C_R(X)$ but is not in $A \oplus B$.

EXAMPLE 2.5. Let $X = [0,1] \times [0,1]$ and let

$$A = \{F \in C_R(X)\colon F(x, y) = f(x) \text{ where } f \in C_R[0,1]\},$$

$$B = \{G \in C_R(X)\colon G(x, y) = g(y) \text{ where } g \in C_R[0,1],\ g(0) = 0\}.$$

Then A and B are subalgebras of $C_R(X)$, $1 \in A$ and $1 \notin B$. The sets of constancy of A are vertical line segments V_x $(0 \leqslant x \leqslant 1)$, where $V_x = \{(x, y)\colon 0 \leqslant y \leqslant 1\}$ and those of B are horizontal line segments W_y $(0 \leqslant y \leqslant 1)$, where $W_y = \{(x, y)\colon 0 \leqslant x \leqslant 1\}$. $Z = W_0 = \{(x, 0)\colon 0 \leqslant x \leqslant 1\}$ and $A \cap B = \{0\}$. A point (x, y) of X is either in Z or there is a chain $(x, y) \overset{A}{-} (x, 0)$ from (x, y) to Z. However the chain is not unique and so $A \oplus B$ is not even dense in $C_R(X)$. In light of Corollary 2.2, $A \oplus B$ is not an algebra.

It is known that if B is a closed ideal and if every set of constancy of A intersects Z in precisely one point, then $A \oplus B = C_R(X)$. The previous example shows that this is not true if B is not an ideal.

Finally we remark that the condition of existence of chains from points of X to Z in the first part of Theorem 2.1 is necessary only for $A \oplus B = C_R(X)$. Examples can be given where $A + B = C_R(X)$ but there is no chain from some points of X to Z.

REFERENCES

1. W. G. Bade, *The Banach space $C(S)$*, Lecture Notes, vol. 26, Aarhus Univ., Aarhus, 1971.

2. S. D. Fisher, *The decomposition of $C_r(K)$ into the direct sum of subalgebras*, J. Funct. Anal. **31** (1979), 218–223.

3. D. E. Marshall and A. G. O'Farrell, *Uniform approximation by real functions*, Fund. Math. **104** (1979), 203–211.

4. Z. Semadeni, *Banach spaces of continuous functions*, Vol. 1, Monografie Mat., Warszawa, 1971.

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