THE EIGENFUNCTIONS OF COMPACT WEIGHTED ENDOMORPHISMS OF \( C(X) \)

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ABSTRACT. In this note we characterize the eigenmanifolds of compact operators \( uC\phi : f \to u \cdot f \circ \phi \) on \( C(X) \) and determine their ascents. As an application we show an easy method for computing the eigenmanifolds of a matrix with at most one nonzero element in each row.

In the sequel \( X \) will always denote a compact Hausdorff space, \( u \) a function in \( C(X) \), and \( \phi \) a continuous function from \( X \) to \( X \). Let \( \phi^n \) be the \( n \)th iterate of \( \phi \); i.e., \( \phi_0(x) = x \) and \( \phi_n(x) = \phi(\phi_{n-1}(x)) \) for \( n > 0 \) and \( x \in X \). \( c \in X \) is called a fixed point of \( \phi \) of order \( n \) if \( n \) is a positive integer, \( \phi_n(c) = c \), and \( \phi_k(c) \neq c \) for \( k = 1, \ldots, n - 1 \).

By \( uC\phi \) we denote the operator \( uC\phi : f \to u \cdot f \circ \phi \) on \( C(X) \). This is a weighted endomorphism, and every weighted endomorphism may be represented in this way (see Kamowitz [1]). Kamowitz [1] proved the following result:

**Theorem A.** Suppose \( X \) is a compact Hausdorff space, \( u \) in \( C(X) \), and \( \phi \) a continuous function from \( X \) into \( X \).

(1) The map \( uC\phi : f \to u \cdot f \circ \phi \) is compact iff for each connected component \( C \) of \( \{x \mid u(x) \neq 0\} \) there exists an open set \( \mathcal{V} \supset C \) such that \( \phi \) is constant on \( \mathcal{V} \).

(2) If \( uC\phi \) is compact, then \( uC\phi \) is compact iff for each connected component \( C \) of \( \{x \mid u(x) \neq 0\} \) there exists an open set \( \mathcal{V} \supset C \) such that \( \phi \) is constant on \( \mathcal{V} \).

Our aim here is to characterize the eigenfunctions of a compact \( uC\phi \). To do that we need some more notation: We always assume that \( \phi \) satisfies the conditions of Theorem A(1) so that \( uC\phi \) is compact. We call \( x, y \in X \) equivalent \((x \sim y)\) if there exist \( n, m \) in \( \{0, 1, 2, \ldots\} \) so that \( \phi_n(x) = y \) and \( \phi_m(y) = x \). The equivalence classes are denoted by \( [x] \). For any \( \lambda \) in \( \mathbb{C} \setminus \{0\} \) let \( C_\lambda := \{c \in X \mid c \text{ is a fixed point of } \phi \text{ of order } n \text{ for some positive integer } n \text{ and } \lambda^n = u(c) \cdots u(\phi_{n-1}(c)) \text{ or } \lambda = 0 \} \).

Obviously if \( x \sim y \) and \( x \in C_\lambda \), then \( y \in C_\lambda \), so let \( \tilde{C}_\lambda := \{[x] \mid [x] \in C_\lambda \} \) and \( m_\lambda \) be the number of equivalence classes in \( \tilde{C}_\lambda \). \( m_\lambda \) is finite by Theorem B and the compactness of \( uC\phi \). For every \( c \in C_\lambda \) let \( h_{c,\lambda} \) denote the following function from \( X \) to \( \mathbb{C} \) or \( \mathbb{R} \) respectively:

\[
h_{c,\lambda}(x) := \begin{cases} \lambda^{-r}u(x) \cdots u(\Phi_r(x)) & \text{for every } r \in \{0, 1, 2, \ldots\} \text{ and } x \in \Phi_r^{-1}(\{c\}), \\ 0 & \text{otherwise.} \end{cases}
\]

It is easy to see that \( h_{c,\lambda} \) is well defined (remember that e.g. \( c \) is in every \( \Phi_{kn}^{-1}(\{c\}) \) if \( c \) is a fixed point of \( \phi \) of order \( n \), but then \( \lambda^{kn} = u(c) \cdots u(\Phi_{kn-1}(c)) \)). Furthermore...
\{h_{c_1,\lambda, \ldots, h_{c_k,\lambda}}\} is linearly dependent iff, for some \(i \neq j\), \(c_i \sim c_j\). Finally, let \(W_0 := W := \{x : u(x) \neq 0\}\) and \(W_k := \Phi(W \cap W_{k-1})\) for \(k > 0\). For additional notation see Taylor [2].

The principal result of this note is the following theorem.

**Theorem B.** (1) Let \(\lambda \in \sigma(uC_\Phi) \setminus \{0\}\) and \(\{c_1, \ldots, c_{m_\lambda}\}\) be representative elements of all equivalence classes in \(C_\lambda\). Then \(\{h_{c_1,\lambda}, \ldots, h_{c_{m_\lambda},\lambda}\}\) is a basis for \(N(\lambda - uC_\Phi)\) and \(\alpha(\lambda - uC_\Phi) = 1\), where \(\alpha(\lambda - uC_\Phi)\) denotes the ascent of \(\lambda - uC_\Phi\).

(2) The case \(\lambda = 0\): If \(n > 0\), then \(N((uC_\Phi)^n) = \{f \in C(X) : f(x) = 0\text{ for every } x \in W_n\}\).

Notice that (1) also states that the functions \(h_{c,\lambda}\) are continuous.

We will break up the proof by proving several propositions.

**Proposition 1.** Let \(\lambda \in \sigma(uC_\Phi) \setminus \{0\}\). Then \(h_{c,\lambda}\) is an eigenfunction for \(\lambda\) for every \(c \in C_\lambda\); that is,

(i) \(\lambda h_{c,\lambda}(x) = u(x)h_{c,\lambda}(\Phi(x))\) for all \(x \in X\),

(ii) \(h_{c,\lambda}\) is continuous.

**Proof.** (i) Let \(x \in X\). If \(x \in \Phi_r^{-1}\{\{c\}\}\) for some \(r > 0\), then

\[\lambda h_{c,\lambda}(x) = u(x)(\lambda^{-(r-1)}u(\Phi(x)) \cdots u(\Phi_r(x))) = u(x)h_{c,\lambda}(\Phi(x)).\]

If \(x \notin \Phi_r^{-1}\{\{c\}\}\) for every \(r > 0\), then the same is true for \(\Phi(x)\), so \(\lambda h_{c,\lambda}(x) = 0 = u(x)h_{c,\lambda}(\Phi(x))\).

(ii) (1) Since \(u\) is continuous, \(B = \{x : |u(x)| \geq |\lambda|\}\) is compact. As \(W\) may be covered with open sets \(V_\beta\), so that \(\Phi\) is constant on each \(V_\beta\), \(\Phi(B)\) is finite, of cardinality \(N\), say. Let \(x \in X\) such that \(h_{c,\lambda}(x) \neq 0\), and \(r\) the minimal number so that \(x \in \Phi_r^{-1}\{\{c\}\}\). Now \(x, \Phi(x), \ldots, \Phi_r(x)\) are distinct, whence

\[|h_{c,\lambda}(x)| = |u(x)/|\lambda| \cdot |u(\Phi(x))/|\lambda| \cdots |u(\Phi_{r-1}(x))/|\lambda| \cdots |u(\Phi_r(x))/|\lambda| |u(c)|\]

\[\leq \max\{1, (||u||_\infty/|\lambda|)^N\} \cdot |u(c)| =: M.\]

Therefore \(h_{c,\lambda}\) is bounded on \(X\).

(2) Let \(x \in X\). If \(u(x) = 0\), then \(h_{c,\lambda}(x) = 0\) and for every \(\varepsilon > 0\) there is a neighborhood \(U\) of \(x\) so that \(|u(y)| < \varepsilon|\lambda|/M\) for every \(y \in U\). Therefore

\[|h_{c,\lambda}(y)| = |\lambda|^{-1}|h_{c,\lambda}(\Phi(x))||u(y)| < \varepsilon\]

for every \(y \in U\) and thus \(h_{c,\lambda}\) is continuous at \(x\). If \(u(x) \neq 0\), then \(\Phi\) is constant on an open neighborhood \(U\) of \(x\) and therefore

\[|h_{c,\lambda}(x) - h_{c,\lambda}(y)| = |\lambda|^{-1}|h_{c,\lambda}(\Phi(x))||u(x) - u(y)|| < \varepsilon\]

for a suitable neighborhood \(U' \subset U\) of \(x\) and every \(y \in U'\). So \(h_{c,\lambda}\) is continuous.

**Proposition 2.** Let \(\lambda \in \sigma(uC_\Phi), \lambda \neq 0\), and \(f\) an eigenfunction for \(\lambda\). Then

(i) For every \(c \in C_\lambda\) there exists \(\alpha(c)\) such that \(f(x) = \alpha(c)h_{c,\lambda}(x)\) for every \(r \geq 0\) and \(x \in \Phi_r^{-1}\{\{c\}\}\).

(ii) If \(x \notin \Phi_r^{-1}\{\{c\}\}\) for every \(c \in C_\lambda\) and \(r \geq 0\), then \(f(x) = 0\).

**Proof.** (i) Let \(c \in C_\lambda\) and \(\alpha(c) := f(c)/u(c)\) (remember \(\lambda \neq 0\)). Then for \(r \geq 0\) and \(x \in \Phi_r^{-1}\{\{c\}\}\) we have by iteration

\[f(x) = \lambda^{-r}u(x)u(\Phi(x)) \cdots u(\Phi_{r-1}(x))f(\Phi_r(x)) = \alpha(c)h_{c,\lambda}(x).\]
(ii) This part of the proof is actually the same as for Proposition 4 in [1] and is repeated here for the sake of completeness:

Let \( x \notin \Phi^{-1}_r(\{c\}) \) for every \( c \in C_\lambda, r \geq 0 \). If \( x \) is a fixed point of \( \Phi \), of order \( n \), say, then by iteration \( f(x) = \lambda^{-n}u(x) \cdots u(\Phi_{n-1}(x))f(x) \) and, since \( x \notin C_\lambda \), we conclude that \( f(x) = 0 \).

If \( x \in \Phi^{-1}_r(\{c\}) \) for some fixed point \( c \in C_\lambda \) and \( r \geq 1 \), then, since \( f(c) = 0 \), we have \( f(x) = \lambda^{-n}u(x) \cdots u(\Phi_{r-1}(x))f(c) = 0 \).

Finally, we may suppose that all \( \Phi_r(x) \) are distinct. Let \( \delta := |\lambda|/2 \). Since \( B := \{x \mid |u(x)| \geq \delta\} \) is compact and by Theorem A \( W \) may be covered by open sets on which \( \Phi \) is constant, \( \Phi(B) \) is finite, of cardinality \( N \), say. Therefore for every \( n > N \)

\[
|f(x)| = |u(x)/\lambda||u(\Phi(x))/\lambda| \cdots |u(\Phi_{n-1}(x))/\lambda||f(\Phi_n(x))| \\
\leq (||u||_{\infty}/|\lambda|)N^2N^{-n}||f||_{\infty} \to 0 \quad (n \to \infty).
\]

Thus \( f(x) = 0 \). Q.E.D.

Let \( \{c_1, \ldots, c_{m_\lambda}\} \) be representative elements of all equivalence classes in \( \tilde{C}_\lambda \). Then \( \{h_{c_1, \lambda}, \ldots, h_{c_{m_\lambda}, \lambda}\} \) is a basis for \( \mathcal{N}(\lambda - A) \) if \( 0 \neq \lambda \in \sigma(uC_\Phi) \). So what remains to be done for part (1) of Theorem B is

**Proposition 3.** Let \( 0 \neq \lambda \in \sigma(uC_\Phi) \) and \( f \in \mathcal{N}((\lambda - uC_\Phi)^2) \). Then \( f \in \mathcal{N}(\lambda - uC_\Phi) \).

**Proof.** Since \( g := (\lambda - uC_\Phi)f \) is an eigenfunction for \( \lambda \), we know by Proposition 2 that if \( x \) is not in \( \Phi^{-1}_r(\{c\}) \) for some \( c \in C_\lambda \) and \( r \geq 0 \), then \( g(x) = 0 \). If \( c \in C_\lambda \) there exists \( \alpha(c) \) so that \( g(x) = \alpha(c)h_{c, \lambda}(x) \) for every \( r \geq 0 \) and \( x \in \Phi^{-1}_r(\{c\}) \) by Proposition 2, so we have to show that \( \alpha(c) = 0 \). Let \( c \) be of order \( n \). Since by iteration

\[
f = n \cdot \frac{g}{\lambda} + (uC_\Phi)^n \cdot \frac{f}{\lambda^n},
\]
evaluation at \( c \) yields

\[f(c) = n\alpha(c)h_{c, \lambda}(c)/\lambda + f(c),\]
for \( g \) is an eigenfunction and \( \lambda^n = u(c) \cdots u(\Phi_n(c)) \). Therefore \( \alpha(c) = 0 \).

So far we have proved Theorem B(1). Part (2) follows from

**Proposition 4.** \((uC_\Phi)^k f = 0 \iff f(x) = 0 \) for every \( x \in W_k \).

**Proof.** By induction:

\((\Rightarrow)\) Let \( k = 1 \) and \( uC_\Phi f = 0 \). Then for any \( x \in W \) we have \( 0 = u(x)f(\Phi(x)) \), whence \( f(\Phi(x)) = 0 \). If \( k > 1 \) and \( (uC_\Phi)^k f = 0 \), we know by induction that \( u(x)f(\Phi(x)) = 0 \) for every \( x \in W_{k-1} \). Furthermore, if \( x \in W \), then \( u(x) \neq 0 \), so that \( f(\Phi(x)) = 0 \). Thus \( f \) vanishes on \( W_k \).

\((\Leftarrow)\) Let \( k = 1 \) and \( f(x) = 0 \) for every \( x \in W_1 \). For \( x \in X \) either \( x \in W \) and therefore \( f(\Phi(x)) = 0 \) or \( x \notin W \) and \( u(x) = 0 \). Thus \( uC_\Phi f = 0 \). Now let \( k > 1 \) and \( f(x) = 0 \) for every \( x \in W_k \). We have to show that \( u(x)f(\Phi(x)) = 0 \) for every \( x \in W_{k-1} \), because then the assertion follows by induction hypothesis. But this is trivial since either \( x \notin W \) and \( u(x) = 0 \), or \( \Phi(x) \in W_k \) and \( f(\Phi(x)) = 0 \), if \( x \in W_{k-1} \).

**Example 1.** We want to give an example for Theorem B(2) that the case \( \mathcal{N}((uC_\Phi)^n) \neq \mathcal{N}((uC_\Phi)^{n+1}) \) for ever \( n \) may occur. Let \( X := \{0\} \cup \{1/n \mid n \in \mathbb{N}\} \) with the topology induced by the usual topology on \( \mathbb{R} \) so that \( X \) is compact. Let
$u(x) = x$ and $\Phi(1/n) = 1/(n+1)$, $\Phi(0) = 0$. These are continuous functions satisfying the conditions of Theorem A. Therefore $uC_\Phi$ is a compact operator on $C(X)$, where $C(X)$ may obviously be identified with $c(N) := \{(a_n)_{n\in\mathbb{N}} | \lim_{n\to\infty} a_n \text{ exists}\}$. Since there are no fixed points $c \neq 0$ of $\Phi$ of any order, $\sigma(uC_\Phi) = \{0\}$ by Theorem A. Now $W_k = \{x \in X | 0 < x < 1/k\}$, so $\mathcal{N}((uC_\Phi)^k) = \{(a_n) | a_n = 0 \text{ for every } n > k\}$ and the union of all $\mathcal{N}((uC_\Phi)^k)$ is exactly the set of all $(a_n)$ satisfying $a_n = 0$ for all but finitely many $n$.

**Example 2.** We give an application of our results to the finite-dimensional case. Let $X = \{1, \ldots, n\}$ with the discrete topology. Then $\mathcal{C}(X)$ will be identified with $K^n$, where $K = \mathbb{C}$ or $K = \mathbb{R}$ is the underlying scalar field. Every linear operator may (and will) be identified with the matrix $(a_{ij})_{1 \leq i, j \leq n}$ with $a_{ij} = (A\delta_j)(i)$, where $\delta_j(i) = 1$, $\delta_j(i) = 0$ if $i \neq j$.

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

$\sigma(uC_\Phi) = \{-2, 0, 2\}$

$\mathcal{N}(A) = \{(x_n) | x_1 = x_2 = x_5 = x_6 = 0\}, \mathcal{N}(A^2) = \{(x_n) | x_5 = x_2 = 0\} = \mathcal{N}(A^3)$.

If $A = uC_\Phi$, then $a_{ij} = u(i)$ if $j = \Phi(i)$ and $a_{ij} = 0$ otherwise, so there is at most one nonzero element in each row. Conversely let $A$ have this property. Then for $i = 1, \ldots, n$ let $j = \Phi(i)$ and $u(i) = a_{ij}$, if $a_{ij}$ is the unique nonzero element in row $i$. If $a_{ij} = 0$ for all $j = 1, \ldots, n$ we let $i = \Phi(i)$ and $u(i) = 0$. Then obviously $A = uC_\Phi$.

Now the eigenvalues and eigenvectors are easily determined: first find out all cycles of $\Phi$, e.g. by drawing $n$ dots with numbers $1, \ldots, n$ and an arrow from dot $j$ to dot $i$ if $\Phi(i) = j$, adding $u(i)$ to that arrow for later purposes. For each cycle multiply all the $u(i)$ of this cycle and calculate the $k$th roots, where $k$ denotes the number of elements of this cycle: these are all eigenvalues possibly except 0.

Take one eigenvalue $\lambda \neq 0$ and a cycle corresponding to that $\lambda$. Choose an arbitrary dot $j$, say, of that cycle and set $x_j := u(j)$. Now follow the arrows. If you reach dot $i$ from dot $k$ let $x_i$ be the product of $\lambda^{-1}u_i$ and $x_k$. When you are done with all the dots which belong to the "connected component" containing the cycle set all other $x_i = 0$. This is an eigenvector for $\lambda$.

If you do this for every cycle corresponding to $\lambda$ you get a basis for the eigenspace $\mathcal{N}(\lambda - A)$.

In order to determine $\mathcal{N}(A^r)$ remove all arrows where $u_i = 0$. Now $\mathcal{N}(A)$ consists of all $(x_k)$, where $x_k = 0$ if there is a directed path of length one starting in dot $k$ (to dot $k$ itself or any other dot), and $x_k$ is arbitrary otherwise. Similarly for $\mathcal{N}(A^r)$, $r > 1$: "one" has to be replaced by "$r$" and it is allowed to "use" the same arrow more than one time.
There is a diagonalization for $A$ iff $\mathcal{N}(A) = \mathcal{N}(A^2)$. Of course all these results are easily obtained by direct verification as well.

REFERENCES


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