A NOTE ON TWIST SPUN KNOTS
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ABSTRACT. A movie presentation for the twist spun knots of an arc is given.

Some time ago Francisco González-Acuña asked me for a movie presentation of the twist spun knots defined and studied by Zeeman [Z]. Since then, other low dimensional topologists have posed the same question to me. Perhaps the following easy solution may have some interest.

**Lemma.** Let $K$ be the knot in plat presentation of Figure 1 where $x$ belongs to the braid group $B_{2m+1}$. Then the $n$-twist spun knot of $K$ is given by the diagram of Figure 2.

**Remark.** In Figure 2 we use Lomonaco’s notation [L]. The diagram of Figure 3(a) is explained in Figure 3(b). The critical saddle point occurs at level $t = \theta$.

![Figure 1](image1.png)  
![Figure 2](image2.png)  
![Figure 3(a)](image3a.png)  
![Figure 3(b)](image3b.png)

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PROOF. By deleting the interior of a regular neighborhood of a point in $K$ we obtain a pair $(B^3, B^3 \cap K)$ which we denote by $(B^3, K)$. Consider the height function $f: B^3 \to [-1, 1]$ of Figure 4.

Let $t \in [0, 1]$ be the parameter measuring the spinning process of $B^3$ so that at time $t = 1$ the ball arrives to its original position. Assume that the twisting of $K$ occurs during the interval $[\frac{1}{2}, 1]$.

We want a movie of the $n$-twist spun knot $K_n$ of $K$ with respect to "hyperplane" sections $S^2_r$, $r \in (-1, 1)$, where $S^2_r$ is the result of spinning the subset $f^{-1}(r)$ of $B^3$. For $r \in \{-1, 1\}$, $f^{-1}(r)$ is just a point.

To achieve this we first define an isotopy of $S^4$ which places the saddle points of $(f \times \text{id})|K_n$ in the level $S^3_0$. This isotopy is defined in three steps.

Step 1. Consider the model halfball $D^3$ and the set $A$ of $m + 1$ arcs shown in Figure 5(a). There is an isotopy $g: A \times I \to D^3$ which pushes $m$ arcs of $A$ onto the boundary. In Figure 5(b) we see the images of $A$ for some values of the parameter.
Step 2. Let $B^3_+$ and $B^3_-$ be the halfballs $f^{-1}[0,1]$ and $f^{-1}[0,-1]$ of $B^3$. Let $F_+$ and $F_-$ be homeomorphisms $F_{\pm} : (D^3, A) \to (B^3_{\pm}, B^3_+ \cap \bar{K})$ and define isotopies $g_{\pm} : \bar{K} \times I \to B^3$ as follows: $g_{\pm}$ is the identity map in $(B^3_{\pm} \cap \bar{K}) \times t$, $t \in I$, and equals $F_{\pm} g F_{\pm}^{-1}$ in $(B^3_{\pm} \cap \bar{K}) \times I$. We embed $g_{\pm}$ in ambient isotopies $G_{\pm} : B^3 \times I \to B^3$. Note that $G_{-}((B^3_+ \cap \bar{K}) \times 1)$ is the set of arcs $b$ together with the point $c$ of Figure 6, if we think of $F_{-}$ as the identity map. Under this condition the set $G_{+}((B^3_+ \cap \bar{K}) \times 1)$
is the image of $b \cup c$ under the action of $x^{-1} \in B_{2m+1}$ on $(f^{-1}(0), f^{-1}(0) \cap K)$. In Figure 7 we show the case $x = \sigma_2^{-1}\sigma_4^{-1}\sigma_3\sigma_2^{-1}$.

Step 3. We now define an isotopy of $S^4$. This isotopy connects the identity map with a map $h: S^4 \to S^4$ defined as follows. The map $h$ realizes $G_+$ when we spin $B^3$ between $t = 0$ and $t = 1/12$, it is constant for $t \in [1/12, 3/12]$, and undoes $G_+$ between $t = 2/12$ and $t = 3/12$. After that, $h$ does $G_-$ in $[3/12, 4/12]$, is constant in $[4/12, 5/12]$ and undoes $G_-$ in $[5/12, 6/12]$. During $[1/2, 1]$ $h$ is the identity map. In Figure 8 we see $h((B^3 \cap K) \times [3/12, 5/12])$.

The knot $h(K_n)$ is ambient isotopic to $K_n$ but all its saddle points with respect to $f \times id$ are at level $S^3$. We only need to understand $S^3 \cap h(K_n)$. The pair $(S^3, S^3 \cap h(K_n))$ is the union of the result of spinning $(f^{-1}(0), f^{-1}(0) \cap K)$ during $t \in [0, 1/2]$, with the result of spinning $x^{-1}(b \cup c)$ during $t \in [1/12, 2/12]$, with the result of spinning $b \cup c$ during $t \in [3/12, 5/12]$, with the result of $n$-twist spinning $(f^{-1}(0), f^{-1}(0) \cap K)$ during $t \in [1/2, 1]$. The picture for $\hat{K}$ given by $x = \sigma_2^{-1}\sigma_4^{-1}\sigma_3\sigma_2^{-1}$ is in Figure 9.

**Figure 10.**

(a) $K$ is a knot  
(b) $-K$ is the mirror image

**Figure 11.**
We have that \((S_0^3, S_0^3 \cap h(K_n))\) is a torus link with \(2m + 1\) trivial components and \(n\)-full twists, together with two sets of bands which correspond to the saddle points. By shrinking the bands with middle lines \(x^{-1}(b \cup c)\) suitably we see that Figure 9 becomes Figure 10.

**COROLLARY.** The torus link \((2m + 1)n, 2m + 1\) is a slice of a trivial knot in \(S^4\). Links of the form depicted in Figure 11(b) have the same property.

**PROOF.** For the first part take \(x \in B_{2m+1}\) such that \(K\) is a trivial knot. For the second part, remember that the 1-twist spun knot of \(K\) is trivial \([Z]\).

**REFERENCES**


(a) \(K\) is a knot

(b) \(-K\) is the mirror image

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