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A NOTE ON TWIST SPUN KNOTS

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ABSTRACT. A movie presentation for the twist spun knots of an arc is given.

Some time ago Francisco González-Acuña asked me for a movie presentation of the twist spun knots defined and studied by Zeeman [**Z**]. Since then, other low dimensional topologists have posed the same question to me. Perhaps the following easy solution may have some interest.

LEMMA. *Let K be the knot in plat presentation of Figure* 1 *where x belongs to the braid group B_{2m+1}. Then the n-twist spun knot of K is given by the diagram of Figure* 2.

REMARK. In Figure 2 we use Lomonaco's notation [**L**]. The diagram of Figure 3(a) is explained in Figure 3(b). The critical saddle point occurs at level $t = \theta$.

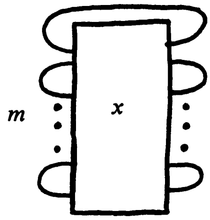

FIGURE 1.

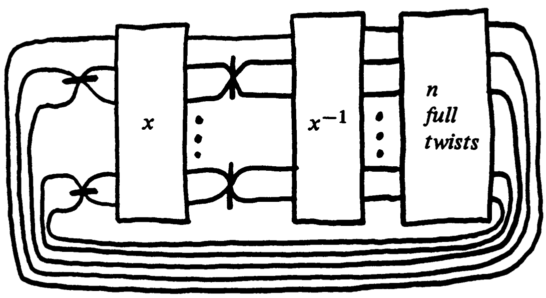

FIGURE 2.

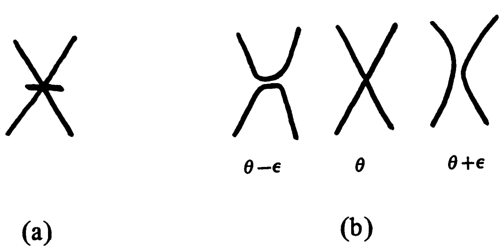

FIGURE 3.

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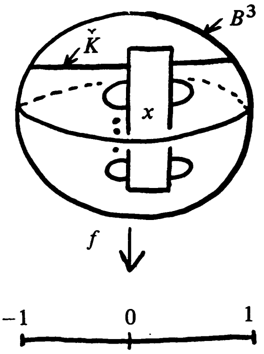


FIGURE 4.

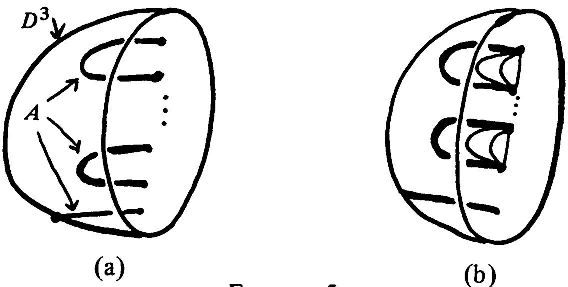


FIGURE 5.

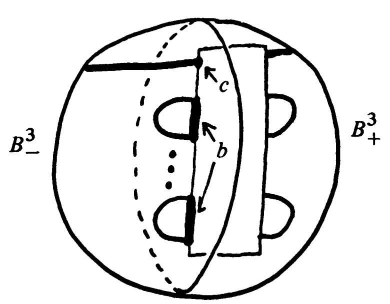


FIGURE 6.

PROOF. By deleting the interior of a regular neighborhood of a point in K we obtain a pair $(B^3, B^3 \cap K)$ which we denote by $(B^3, \check{K})$. Consider the height function $f\colon B^3 \to [-1,1]$ of Figure 4.

Let $t \in [0,1]$ be the parameter measuring the spinning process of B^3 so that at time $t = 1$ the ball arrives to its original position. Assume that the twisting of $\check{K}$ occurs during the interval $[\frac{1}{2}, 1]$.

We want a movie of the n-twist spun knot K_n of K with respect to "hyperplane" sections S^3_r, $r \in (-1,1)$, where S^3_r is the result of spinning the subset $f^{-1}(r)$ of B^3. For $r \in \{-1,1\}$, $f^{-1}(r)$ is just a point.

To achieve this we first define an isotopy of S^4 which places the saddle points of $(f \times \mathrm{id})|K_n$ in the level S^3_0. This isotopy is defined in three steps.

Step 1. Consider the model halfball D^3 and the set A of $m+1$ arcs shown in Figure 5(a). There is an isotopy $g\colon A \times I \to D^3$ which pushes m arcs of A onto the boundary. In Figure 5(b) we see the images of A for some values of the parameter.

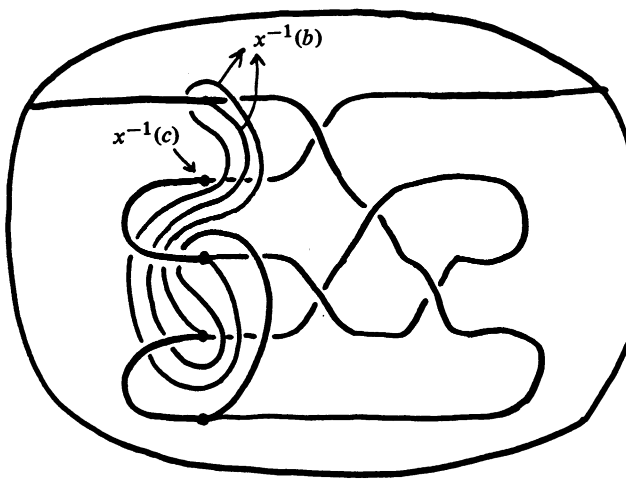


FIGURE 7.

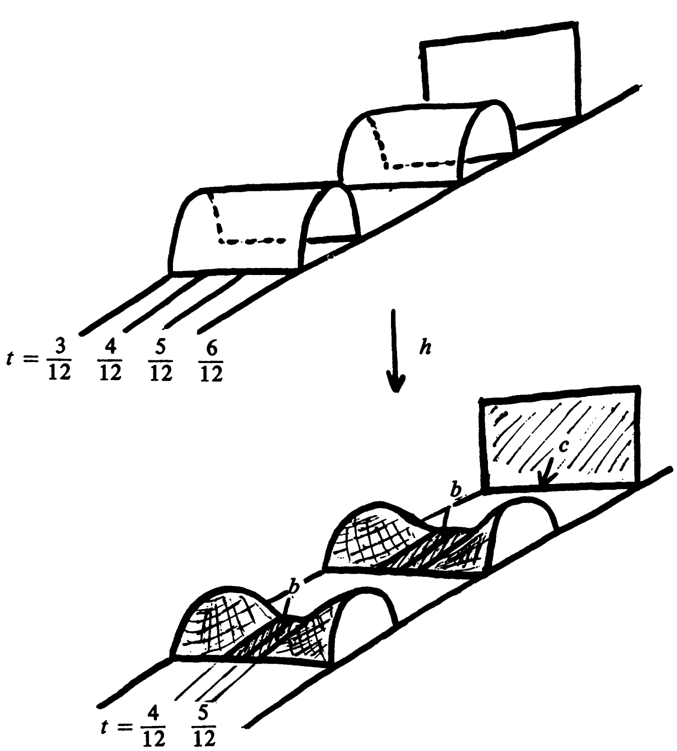


FIGURE 8.

Step 2. Let B^3_+ and B^3_- be the halfballs $f^{-1}[0,1]$ and $f^{-1}[0,-1]$ of B^3. Let F_+ and F_- be homeomorphisms $F_\pm\colon (D^3, A) \to (B^3_\pm, B^3_\pm \cap \check{K})$ and define isotopies $g_\pm\colon \check{K} \times I \to B^3$ as follows: $g_\pm$ is the identity map in $(B^3_\mp \cap \check{K}) \times t$, $t \in I$, and equals $F_\pm g F_\pm^{-1}$ in $(B^3_\pm \cap \check{K}) \times I$. We embed $g_\pm$ in ambient isotopies $G_\pm\colon B^3 \times I \to B^3$. Note that $G_-((B^3_- \cap \check{K}) \times 1)$ is the set of arcs b together with the point c of Figure 6, if we think of F_- as the identity map. Under this condition the set $G_+((B^3_+ \cap \check{K}) \times 1)$

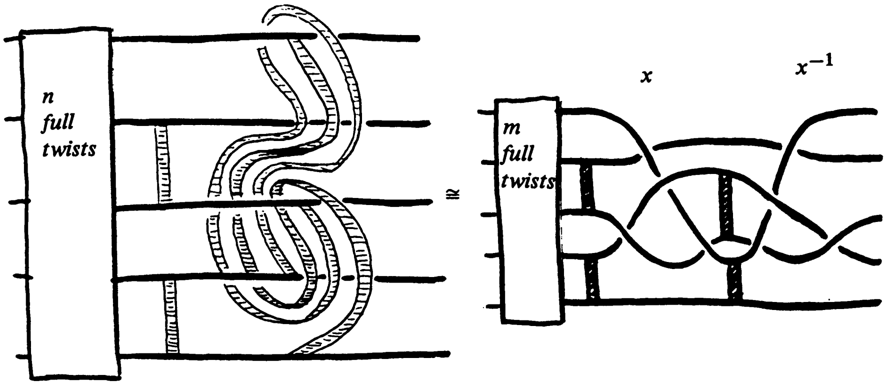


FIGURE 9.

is the image of $b \cup c$ under the action of $x^{-1} \in B_{2m+1}$ on $(f^{-1}(0), f^{-1}(0) \cap \check{K})$. In Figure 7 we show the case $x = \sigma_2^{-1}\sigma_4^{-1}\sigma_3\sigma_2^{-1}$.

Step 3. We now define an isotopy of S^4. This isotopy connects the identity map with a map h: $S^4 \to S^4$ defined as follows. The map h realizes G_+ when we spin B^3 between $t = 0$ and $t = 1/12$, it is constant for $t \in [\frac{1}{12}, \frac{2}{12}]$, and undoes G_+ between $t = 2/12$ and $t = 3/12$. After that, h does G_- in $[\frac{3}{12}, \frac{4}{12}]$, is constant in $[\frac{4}{12}, \frac{5}{12}]$ and undoes G_- in $[\frac{5}{12}, \frac{6}{12}]$. During $[\frac{1}{2}, 1]$ h is the identity map. In Figure 8 we see $h((B^3_- \cap \check{K}) \times [\frac{3}{12}, \frac{6}{12}])$.

The knot $h(K_n)$ is ambient isotopic to K_n but all its saddle points with respect to $f \times \mathrm{id}$ are at level S^3_0. We only need to understant $S^3_0 \cap h(K_n)$. The pair $(S^3_0, S^3_0 \cap h(K_n))$ is the union of the result of spinning $(f^{-1}(0), f^{-1}(0) \cap \check{K})$ during $t \in [0, \frac{1}{2}]$, with the result of spinning $x^{-1}(b \cup c)$ during $t \in [\frac{1}{12}, \frac{2}{12}]$, with the result of spinning $b \cup c$ during $t \in [\frac{4}{12}, \frac{5}{12}]$, with the result of n-twist spinning $(f^{-1}(0), f^{-1}(0) \cap \check{K})$ during $t \in [\frac{1}{2}, 1]$. The picture for $\check{K}$ given by $x = \sigma_2^{-1}\sigma_4^{-1}\sigma_3\sigma_2^{-1}$ is in Figure 9.

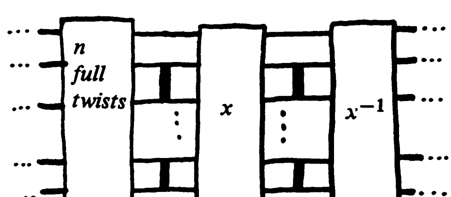
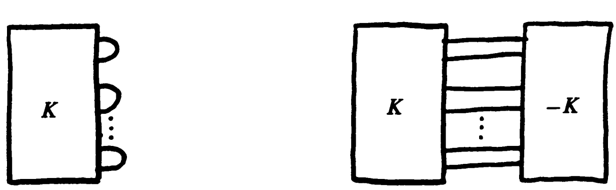


FIGURE 10.



(a) K is a knot (b) $-K$ is the mirror image

FIGURE 11.

We have that $(S_0^3, S_0^3 \cap h(K_n))$ is a torus link with $2m+1$ trivial components and n-full twists, together with two sets of bands which correspond to the saddle points. By shrinking the bands with middle lines $x^{-1}(b \cup c)$ suitably we see that Figure 9 becomes Figure 10.

COROLLARY. *The torus link* $\{(2m+1)n, 2m+1\}$ *is a slice of a trivial knot in* S^4. *Links of the form depicted in Figure* 11(b) *have the same property.*

PROOF. For the first part take $x \in B_{2m+1}$ such that K is a trivial knot. For the second part, remember that the 1-twist spun knot of K is trivial [**Z**].

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(a) K is a knot
(b) $-K$ is the mirror image

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