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ON A QUESTION OF FAITH IN COMMUTATIVE ENDOMORPHISM RINGS

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ABSTRACT. Given a commutative ring R, let $Q(R)$ denote its maximal ring of quotients and, for any ideal I of R, let $\mathrm{End}(I)$ denote the ring of R-endomorphisms of I. It is known that if $Q(R)$ is a self-injective ring then $\mathrm{End}(I)$ is commutative for each ideal I of R. Carl Faith has asked if the converse holds. It does if R is either Noetherian or has no nontrivial nilpotent elements but here we produce an example to show that it does not hold in general.

Introduction. Throughout this paper R is a commutative ring with identity and $Q(R)$ will denote its maximal ring of quotients. The definition and fundamental properties of $Q(R)$ are to be found in, for example, the books of Lambek [**7**] and Stenström [**9**]. In particular, Theorem 4.1 on p. 279 of [**9**] shows that $Q(R)$ is a self-injective ring if and only if for every ideal I of R and every R-homomorphism $\phi\colon I \to R$ there exists a faithful ideal J of R and an R-homomorphism $\psi\colon J \to R$ such that $I \subseteq J$ and ψ extends ϕ.

For every ideal I of R we let $\mathrm{End}(I)$ denote the ring of R-endomorphisms of I. In Proposition 1.2 of [**3**], Cox showed that $\mathrm{End}(J)$ is commutative whenever J is a faithful ideal. It then follows easily from the previous paragraph that if $Q(R)$ is a self-injective ring then $\mathrm{End}(I)$ is commutative for every ideal I of R. Carl Faith has asked, [**4**, p. 199, problem 10] and [**5**, p. 98, problem 6], if the converse holds. Alamelu, [**1** and **2**], and Cox [**3**] have shown that it does with some restrictions on R, in particular, if R is either Noetherian or has no nontrivial nilpotent elements. We produce below an example of a ring R which gives a negative answer to Faith's general question.

The example. Let A be a discrete valuation ring with maximal ideal At and quotient field K. Moreover, assume that A is not complete, for example A can be taken to be countable. Let M denote the A-module K/A and for each $n \in \mathbf{N}$ let M_n denote the A-submodule At^{-n}/A of M where $At^{-n} = \{at^{-n} \colon a \in A\} = \{ut^k \colon u$ is a unit in $A, k \in \mathbf{Z},\ k \geq -n\}$. Then M is the direct union $\bigcup_{n\in\mathbf{N}} M_n$ and any proper nonzero A-submodule of M is M_n for some $n \in \mathbf{N}$.

For each $n \in \mathbf{N}$ let E_n denote the A-endomorphism ring of M_n and let E denote the A-endomorphism ring of M. Any f in E_n is determined by $f(t^{-n} + A)$, which is of the form $a(t^{-n} + A)$ for some a in A. This produces a ring isomorphism $\phi\colon E_n \to A/At^n$ given by $\phi(f) = a + At^n$.

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If x is an arbitrary nonzero element of some M_n we can write $x = ut^{m-n} + A$ where u is a unit in A and $m < n$ Then if $g\colon M_n \to M$ is an A-homomorphism with $g(x) = vt^{-k} + A$, where v is a unit and $k \geq 0$, we have

$$0 = g(t^{n-m}x) = t^{n-m}g(x) = vt^{n-m-k} + A$$

and so $-k \geq m - n$. Hence $g(x) \in M_{n-m} \subseteq M_n$. This argument allows us to define, for any $m, n \in \mathbf{N}$ with $m \leq n$, the restriction functions $\phi_{mn}\colon E_m \to E_n$ by $\phi_{mn}(g) = g|_{M_n}$ and $\phi_n\colon E \to E_n$ defined similarly. This gives a projective system $(E_m, \phi_{mn})_{m,n\in\mathbf{N}}$ of rings and ring homomorphisms and it is readily checked that

$$E = \operatorname{proj\,lim} E_m.$$

Since, from above, each E_n is isomorphic to the commutative ring A/At^n, it follows that E is a commutative ring (see, for example, Proposition 1.9 on p. 53 of Lafon [**6**]).

We now let R be the trivial extension of A by M. In other words, $R = A \oplus M$, made into a ring by defining addition componentwise and multiplication by

$$(a_1 + m_1)(a_2 + m_2) = a_1a_2 + a_1m_2 + a_2m_1$$

for all $a_1, a_2 \in A$, $m_1, m_2 \in M$. This procedure was called *the principle of idealization* by Nagata [**8**] since the ring R contains the ring A and M is an ideal of R, with $M^2 = 0$. We determine the ideals of R.

First, if S is any A-submodule of M, then, by our multiplication in R, S is an ideal of R. Moreover, any ideal of R contained in M is such a submodule. Thus the ideals of R contained in M are precisely 0, M and the M_n. Furthermore, each M_n is just the principal ideal of R generated by $t^{-n} + A$. On the other hand, if r is an element of R not in M, say $r = ut^n + m$, where u is a unit in A, $n \geq 0$ and $m \in M$, then by the divisibility of the A-module $M = K/A$ we have $t^nM = M$. Using this, it is not difficult to show that the principal ideal Rr is $At^n \oplus M$ and coincides with Rt^n. In particular, we have that all ideals of R not contained in M are principal. Thus all the ideals of R except M are principal.

Now from the definition of R it is easily shown that the R-endomorphism ring of M coincides with its A-endomorphism ring, in other words $\operatorname{End}_R(M) = E$. In particular, from above, the R-ideal M has a commutative endomorphism ring. Since all the other ideals of R are principal, they also have commutative endomorphism rings. Thus to establish our example it remains to show that $Q(R)$ is not self-injective.

In fact, $Q(R) = R$. To see this we first note that for any ideal I of R contained in M we have $MI = 0$, while if J is a proper ideal of R not contained in M then $J = At^n \oplus M$ for some $n > 0$ and so $M_nJ = 0$. Thus the only faithful ideal in R is R itself. That $Q(R) = R$ now follows from Proposition 4 on p. 39 of Lambek [**7**].

Now let $\hat{A}$ denote the completion of A and choose an element b belonging to $\hat{A}$ but not to A. Then b can be represented by a formal power series

$$b = u_0 + u_1 + \cdots + u_nt^n + \cdots$$

where the u_i are elements of A coming from a fixed set of representatives for A modulo At and, since b is not in A, u_i is nonzero for infinitely many i. Then multiplication by b gives an R-endomorphism h of the ideal M which differs from any multiplication of M by an element from A. However, for any R-endomorphism

$k\colon R \to R$, if $k(1) = a + m$ where $a \in A$, $m \in M$, then the restriction of k to M is just multiplication by a. This shows that our homomorphism $h\colon M \to R$ does not extend to R. Hence R, and so $Q(R)$, is not self-injective.

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