PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 98, Number 2, October 1986

GENUS GROUP OF FINITE GALOIS EXTENSIONS

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ABSTRACT. Let K/k be a Galois extension of finite degree, and let K' denote the maximal abelian extension over k contained in the Hilbert class field of K. We give formulas about the group structure of $\mathrm{Gal}(K'/k)$ and the genus group of K/k, which refine the ordinary genus formula.

For an algebraic number field k of finite degree, let $\mathrm{Cl}(k)$ denote the ideal class group of k. For a modulus S of k (i.e., a finite product of primes of k), let $I_k(S)$, $P_k(S)$, and P_{kS} denote the group of ideals of k prime to S, the group of principal ideals of k prime to S, and the ray ideal group modulo S in k, respectively. Similarly, let $k(S)$ and k_S denote the group of elements of k prime to S and the ray number group of k modulo S, respectively. Let K/k be a Galois extension of finite degree. Let K' be the maximal abelian extension over k contained in the Hilbert class field $\overline{K}$ of K. Then, by definition, the genus field K^* of K/k in the wide sense is $K \cdot K'$, and the genus group of K/k is $\mathrm{Gal}(K^*/K)$.

The following lemma is well known and proved by a standard manner in class field theory.

LEMMA 1. *Let $\mathfrak{f}'$ be the conductor of K'/k; then K' is a class field over k corresponding to $I_k(\mathfrak{f}')/N_{K/k}(P_K(\mathfrak{f}'))P_{k\mathfrak{f}'}$. Moreover, if K/k is abelian, then $\mathfrak{f}'$ coincides with the conductor of K/k.*

For a modulus $\mathfrak{f}$ with $\mathfrak{f}' \mid \mathfrak{f}$, put $\mathfrak{G}(K/k) = I_k(\mathfrak{f})/N_{K/k}(P_K(\mathfrak{f}))P_{k\mathfrak{f}}$. Clearly, $\mathfrak{G}(K/k)$ does not depend on the choice of such $\mathfrak{f}$ (up to isomorphisms). The purpose of this paper is to describe the l^i-rank of $\mathfrak{G}(K/k)$. Let l be a prime number. Throughout this paper we fix l unless otherwise stated. For an abelian group A written multiplicatively, let $\mathrm{rank}_i(A)$ denote the l^i-rank of A, i.e., the F_l-dimension of $A^{l^{i-1}}/A^{l^i}$. For $i \geqslant 0$, put $F_i = \{a \in k^\times \mid (a) \in P_k^{l^i}\}$. Then $k^\times \supset F_1 \supset F_2 \supset \cdots \supset F_i \supset \cdots \supset E_k$ and $F_i \supset F_{i-1}^l E_k$, where E_k denotes the group of units in k. Put $F_i(S) = F_i \cap k(S)$.

LEMMA 2. *Let $1 \to N \to M \to L \to 1$ $(N \subset M)$ be an exact sequence of finite abelian groups. Put $N_i = N \cap M^{l^i}$. Then for $i \geqslant 1$, we have*

$$\begin{aligned}\mathrm{rank}_i(M) &= \mathrm{rank}_i(L) + \mathrm{rank}_1(N_{i-1}/N_i)\\ &= \mathrm{rank}_i(L) + \log_l\{\#(N_{i-1}/N_i)\}.\end{aligned}$$

Received by the editors October 11, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 11R37, 11R65.

PROOF. By the definition of l^i-rank, we see $\mathrm{rank}_i(L) = \mathrm{rank}_1(L^{l^{i-1}}/L^{l^i}) = \mathrm{rank}_1(M^{l^{i-1}}N/M^{l^i}N)$. On the other hand,

$$\begin{aligned}\#\left(M^{l^{i-1}}N/M^{l^i}N\right) &= \#\left(M^{l^{i-1}}/M^{l^{i-1}} \cap N\right)/\#\left(M^{l^i}/M^{l^i} \cap N\right)\\ &= \#\left(M^{l^{i-1}}/M^{l^i}\right)/\#(N_{i-1}/N_i).\end{aligned}$$

This proves the lemma.

PROPOSITION 1. *Put* $\mathscr{N}_i(\mathfrak{f}) = k(\mathfrak{f})^{l^i}N_{K/k}(K(\mathfrak{f}))k_{\mathfrak{f}}$. *Then*

$$\begin{aligned}\mathrm{rank}_i(\mathfrak{G}(K/k)) &= \mathrm{rank}_i(\mathrm{Cl}(k)) + \mathrm{rank}_i\left(k(\mathfrak{f})/N_{K/k}(K(\mathfrak{f}))k_{\mathfrak{f}}\right)\\ &\quad + \log_l\{\#(F_{i-1}(\mathfrak{f})/F_{i-1}(\mathfrak{f}) \cap \mathscr{N}_{i-1}(\mathfrak{f}))/\#(F_i(\mathfrak{f})/F_i(\mathfrak{f}) \cap \mathscr{N}_i(\mathfrak{f}))\}.\end{aligned}$$

PROOF. We apply Lemma 2 to an exact sequence

$$\begin{aligned}1 \to P_k(\mathfrak{f})/N_{K/k}(P_K(\mathfrak{f}))P_{k\mathfrak{f}} &\to I_k(\mathfrak{f})/N_{K/k}(P_K(\mathfrak{f}))P_{k\mathfrak{f}}\\ &\to I_k(\mathfrak{f})/P_k(\mathfrak{f}) \to 1.\end{aligned}$$

Then with the notations in Lemma 2 we have

$$\begin{aligned}N_i &= \left(P_k(F) \cap I_k(\mathfrak{f})^{l^i}N_{K/k}(P_K(\mathfrak{f}))P_{k\mathfrak{f}}\right)/N_{K/k}(P_K(\mathfrak{f}))P_{k\mathfrak{f}}\\ &\cong F_i(\mathfrak{f})N_{K/k}(K(\mathfrak{f}))k_{\mathfrak{f}}/E_kN_{K/k}(K(\mathfrak{f}))k_{\mathfrak{f}}.\end{aligned}$$

Hence

$$N_{i-1}/N_i \cong F_{i-1}(\mathfrak{f})N_{K/k}(K(\mathfrak{f}))k_{\mathfrak{f}}/F_i(\mathfrak{f})N_{K/k}(K(\mathfrak{f}))k_{\mathfrak{f}}.$$

So

$$\begin{aligned}\#(N_{i-1}/N_i) &= \left\{\frac{\#(F_{i-1}(\mathfrak{f})N_{K/k}(K(\mathfrak{f}))k_{\mathfrak{f}}/\mathscr{N}_{i-1}(\mathfrak{f}))}{\#(F_i(\mathfrak{f})N_{K/k}(K(\mathfrak{f}))k_{\mathfrak{f}}/\mathscr{N}_i(\mathfrak{f}))}\right\}\#(\mathscr{N}_{i-1}(\mathfrak{f})/\mathscr{N}_i(\mathfrak{f}))\\ &= \left\{\frac{\#(F_{i-1}(\mathfrak{f})/F_{i-1}(\mathfrak{f}) \cap \mathscr{N}_{i-1}(\mathfrak{f}))}{\#(F_i(\mathfrak{f})/F_i(\mathfrak{f}) \cap \mathscr{N}_i(\mathfrak{f}))}\right\}\#(\mathscr{N}_{i-1}(\mathfrak{f})/\mathscr{N}_i(\mathfrak{f})).\end{aligned}$$

Thus Lemma 2 implies the assertion.

Let v be a prime of k ramified in K, and let V be a prime divisor of v in $\overline{K}$. We also denote the restriction of V to an intermediate field of $\overline{K}/k$. Let $\overline{K}_V$, K_V, K'_V, and k_v denote the completion of $\overline{K}, K, K'$, and k by V, respectively. Further, let $(K_V)''$ be the maximal abelian subextension of $\overline{K}_V/k_v$, then $K'_V \subset (K_V)''$. Moreover, let $(K_V)_2$ be the maximal abelian subextension of K_V/k_v, and let $\mathfrak{f}_V$ and T_V denote the conductor of $(K_V)_2/k_v$ and the inertia group of v in $(K_V)_2/k_v$, respectively. Since K/k is Galois, the conductor $\mathfrak{f}_V$ and the group T_V do not depend on the choice of a prime divisor of v. Therefore, we write $\mathfrak{f}_v$ and T_v instead of $\mathfrak{f}_V$ and T_V. On the other hand, the conductor of $(K_V)''/k_v$ coincides with $\mathfrak{f}_v$ since $N_{\overline{K}_V/k_v}(U(\overline{K}_V)) = N_{K_V/k_v}(U(K_V))$, where $U(K_V)$ denotes the group of units of K_V. Thus if we put $\mathfrak{f}^* = \prod_v \mathfrak{f}_v$, then $\mathfrak{f}' \mid \mathfrak{f}^*$, so we can apply the above results to $\mathfrak{f}^*$.

THEOREM. *Let the notation be as above. For $i \geqslant 1$, we have*

$$\operatorname{rank}_i(\mathfrak{G}(K/k)) = \operatorname{rank}_i(\mathrm{Cl}(k)) + \sum_v \operatorname{rank}_i(T_v) + \log_l\left\{\frac{\#(F_{i-1}(\mathfrak{f}^*)/F_{i-1}(\mathfrak{f}^*) \cap \mathcal{N}_{i-1}(\mathfrak{f}^*))}{\#(F_i(\mathfrak{f}^*)/F_i(\mathfrak{f}^*) \cap \mathcal{N}_i(\mathfrak{f}^*))}\right\}.$$

Moreover, if K/k is abelian, then $\mathfrak{f}^*$ is the conductor of K/k and T_v is the inertia group of v in K/k.

PROOF. We apply Proposition 1 for $\mathfrak{f} = \mathfrak{f}^*$. Then it suffices to prove the assertion about the second term in the right-hand side of the above formula. Clearly,

$$k(\mathfrak{f})/N_{K/k}(K(\mathfrak{f}))k_{\mathfrak{f}} = \prod_v \left(k(\mathfrak{f}_v)/N_{K/k}(K(\mathfrak{f}_v))k_{\mathfrak{f}_v}\right) \quad \text{(direct)}$$

holds. Further, $k_{\mathfrak{f}_v} \subset N_{K_V/k_v}(U(K_V))$ since $\mathfrak{f}_v$ is the conductor of $(K_V)''/k_v$. So we have a natural homomorphism: $k(\mathfrak{f}_v)/N_{K/k}(K(\mathfrak{f}_v))k_{\mathfrak{f}_v} \to U(k_v)/N_{K_V/k_v}(U(K_V))$; but noting K/k is Galois, we can easily check that this gives an isomorphism. On the other hand, local class field theory states $U(k_v)/N_{K_V/k_v}(U(K_V)) \cong T_v$, which proves the theorem.

Here the last term of the above is rewritten as

$$\log_l\left\{\frac{\left[\#\left(F_{i-1}(\mathfrak{f})/k(\mathfrak{f})^{l^{i-1}}\right)/\#\left(F_{i-1}(\mathfrak{f}) \cap \mathcal{N}_{i-1}(\mathfrak{f})/k(\mathfrak{f})^{l^{i-1}}\right)\right]}{\left[\#\left(F_i(\mathfrak{f})/k(\mathfrak{f})^{l^i}\right)/\#\left(F_i(\mathfrak{f}) \cap \mathcal{N}_i(\mathfrak{f})/k(\mathfrak{f})^{l^i}\right)\right]}\right\};$$

hence from [**3**, Lemma 1] we have the following:

COROLLARY.

$$\operatorname{rank}_i(\mathfrak{G}(K/k)) = \sum_v \operatorname{rank}_i(T_v) - \operatorname{rank}_i(E_k) + \log_l\left\{\frac{\#\left(F_i(\mathfrak{f}^*) \cap \mathcal{N}_i(\mathfrak{f}^*)/k(\mathfrak{f}^*)^{l^i}\right)}{\#\left(F_{i-1}(\mathfrak{f}^*) \cap \mathcal{N}_{i-1}(\mathfrak{f}^*)/k(\mathfrak{f}^*)^{l^{i-1}}\right)}\right\}.$$

REMARK 1. We know for sufficiently large i and j (independent of l), $F_i(\mathfrak{f}^*)/F_i(\mathfrak{f}^*) \cap \mathcal{N}_i(\mathfrak{f}^*) \cong E_k/E_k \cap \mathcal{N}_i(\mathfrak{f}^*)$ and $E_k \cap \mathcal{N}_i(\mathfrak{f}^*) = E_k \cap \mathcal{N}_j(\mathfrak{f}^*)$. Moreover, taking a product for all l, we have $E_k/E_k \cap N_{K/k}(K(\mathfrak{f}^*))k_{\mathfrak{f}^*} \cong \prod_l(E_k/E_k \cap \mathcal{N}^{(l)}(\mathfrak{f}^*))$, where $\mathcal{N}^{(l)}$ denotes $\mathcal{N}_i$ corresponding to l. Thus multiplying the formulas for all l and i in the above theorem, we have

$$\#(\mathfrak{G}(K/k)) = h(k) \cdot \prod_v \#(T_v)/\left[E_k : E_k \cap N_{K/k}(K(\mathfrak{f}^*))k_{\mathfrak{f}^*}\right].$$

Let K_1 denote the maximal abelian subextension of K/k. Then $[K^* : K] = [K' : K_1] = \#(\mathfrak{G}(K/k))/[K_1 : k]$, so the above is nothing but the genus formula (e.g. see [**1**]). Thus the above theorem refines the genus formula.

REMARK 2. Now we consider an abelian case. Let T be a finite set of primes of k. Let M_i be the maximal abelian extension at most of index l^i in which only primes in T are ramified. For the conductor $\mathfrak{f}$ of M_i, let M be the ray class field modulo $\mathfrak{f}$. Then M_i is the maximal subfield of M at most of index l^i. Hence $\log_l \#(\mathrm{Gal}(M_i/k)) = \sum_{j=1}^{i} \mathrm{rank}_j\, \mathfrak{G}(M/k)$. Put $h_i = \#\{x \in \mathrm{Cl}(k) \mid x^{l^i} = 1\}$ and $t_i(v) = \#\{x \in T_v \mid x^{l^i} = 1\}$. Then the above theorem implies

$$\#(\mathrm{Gal}(M_i/k)) = h_i(k) \prod_{v \in T} t_i(v) / [F_i(\mathfrak{f}) : F_i(\mathfrak{f}) \cap \mathcal{N}_i(\mathfrak{f})].$$

Since M is the ray class field, $\mathcal{N}_i(\mathfrak{f}) = k(\mathfrak{f})^{l^i} k_{\mathfrak{f}}$. Thus the above theorem gives a generalization of Kubota-Miki's formula [**3**, Theorem 1] (cf. [**2**]). Of course, in this case if $i = 1$, then the corollary to the above theorem is [**4**, Theorem 1].

Finally we study the genus group of K/k. In general, the genus group is not determined only by $\mathfrak{G}(K/k)$ and $\mathrm{Gal}(K/k)$. Indeed, as is easily seen there are abelian fields K/Q and L/Q such that $\mathrm{Gal}(K/Q) \cong \mathrm{Gal}(L/Q)$ and $K^* = L^*$ although $\mathrm{Gal}(K^*/K)$ is not isomorphic to $\mathrm{Gal}(L^*/L)$. However, in the following case the genus group is completely determined by $\mathfrak{G}(K/k)$ and $\mathrm{Gal}(K_1/k)$ since $\mathrm{Gal}(K^*/K) \cong \mathrm{Gal}(K'/K_1)$. The proof of Proposition 2 is easy, so we omit it.

PROPOSITION 2. *If there exists a finite set T of primes of k such that* $\mathrm{Gal}(K_1/k) = \prod_{v \in T} T_v$ *(direct product), then*

$$\mathfrak{G}(K/k) \cong \mathrm{Gal}(K_1/k) \oplus \mathrm{Gal}(K'/K_1).$$

REMARK 3. In this paper we deal with the wide sense, but a similar argument holds in the narrow sense with a few changes of parts about infinite primes.

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