

MEASURABILITY PROPERTIES OF SPECTRA

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ABSTRACT. We study Borel measurability of the spectrum in topological algebras. We give some equivalences of the various properties, show that the spectrum in a Banach algebra is continuous on a dense G_δ , and prove that in a Polish algebra the set of invertible elements is an $F_{\sigma\delta}$ and the inverse mapping is a Borel function of the second class.

This article has its origin in the papers [7] and [5]. We study Borel measurability of the spectrum and related sets and mappings in various classes of algebras.

The best-known example is the case of Banach algebras: the spectrum is then an upper semicontinuous mapping, the set of invertible elements open, and the inverse mapping continuous. In the first part of the paper we establish relations between various measurability properties and study the set of points of measurability of the spectrum considered as a set-valued function. (This part of the paper—in particular Lemma 6 through Corollary 9—can be read independently.) We prove, among other things, the following fact.

THEOREM (CF. THEOREM 7). *If X is a Banach algebra, then the spectrum $x \rightarrow \sigma(x)$: $X \rightarrow 2^{\mathbb{C}}$ is continuous on a dense G_δ in X .*

The second part of the paper uses topological methods to deduce measurability results: we prove that in a Polish algebra the set of invertible elements is an $F_{\sigma\delta}$ and the inverse mapping is a Borel measurable function of the second class.

We prove in particular (Theorem 13), without any assumption of separability or local convexity, that the spectrum of a continuous linear operator acting on a complete metrizable vector space is always a $G_{\delta\sigma}$ set. This answers a question posed by A. L. Shields [6].

Terminology. Throughout the paper X will denote a complex algebra with identity e (commutativity is not assumed), which is also a topological vector space.

The continuity properties of multiplication will be specified case by case.

An F -space is a complete metric vector space.

A Polish space is a complete separable metric space.

A topological space Z is said to be a Baire space if every nonempty open subset of Z is of second category (i.e. cannot be represented as the union of a countable family of nowhere dense subsets of Z).

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Let Y and Z be topological spaces and $B: Y \rightarrow 2^Z$ a set-valued mapping. The graph of B is

$$\text{Gr } B = \{(y, z) \in Y \times Z: z \in B(y)\}$$

and if $V \subset Z$, $B^{-1}(V) = \{y \in Y: B(y) \cap V \neq \emptyset\}$.

B is lower semicontinuous (lsc) at $y \in Y$ if for every V open in Z such that $V \cap B(y) \neq \emptyset$, $y \in \text{int } B^{-1}(V)$.

B is upper semicontinuous (usc) at $y \in Y$ if for every V open in Z such that $B(y) \subset V$, there exists a neighborhood U of y with $B(t) \subset V$ for each t in U .

B is continuous if it is both lsc and usc.

The maps Π_Y and Π_Z on the product $Y \times Z$ denote the natural projections.

The complement of a set A is denoted by CA .

The inverse mapping in the algebra X is denoted by $x \rightarrow x^{-1}$.

Let us recall that X is a complex algebra and a topological vector space, W is the set of invertible elements of X , and $\sigma: X \rightarrow 2^{\mathbb{C}}$ is the spectrum mapping: $\sigma(x) = \{\lambda \in \mathbb{C}: x - \lambda e \notin W\}$.

THEOREM 1. *The following conditions are equivalent:*

- (i) W is open.
- (ii) $\text{Gr } \sigma$ is closed in $X \times \mathbb{C}$.
- (iii) σ is upper semicontinuous.

PROOF. Define $\theta: X \times \mathbb{C} \rightarrow X$ by $\theta(x, \lambda) = x - \lambda e$; θ is continuous, and the equivalence of (i) and (ii) follows from the equalities

$$\theta^{-1}(W) = C \text{Gr } \sigma \quad \text{and} \quad W = \Pi_X[\theta^{-1}(W) \cap (X \times \{0\})].$$

If σ is usc, then W is open because $CW = \sigma^{-1}(\{0\})$.

Suppose, conversely, that W is open; then $V = \{x \in X: \sigma(x) \subset \mathbb{C} \setminus \{1\}\} = W + e$ is an open neighborhood of 0 in X .

Since the map $(x, \lambda) \rightarrow \lambda x$ ($x \in X, \lambda \in \mathbb{C}$) is continuous, there exist $\varepsilon > 0$ and a neighborhood D of 0 in X such that $\bar{B}(0, \varepsilon) \times D \subset V$, where $\bar{B}(0, \varepsilon) = \{t \in \mathbb{C}: |t| \leq \varepsilon\}$. Fix x in D : then for every $|t| \leq \varepsilon$, $t \neq 0$, $1 \notin \sigma(tx)$; that is, $1/t \notin \sigma(x)$ and $\sigma(x) \subset B(0, 1/\varepsilon)$. Thus σ is uniformly bounded on a neighborhood of zero.

Suppose σ is not usc at a point $x_0 \in X$: then there exist an open neighborhood G of $\sigma(x_0)$ in \mathbb{C} and nets $x_j \rightarrow x_0$ and $\lambda_j \in \sigma(x_j) \setminus G$. Pick n in \mathbb{N} such that $x_0 \in nD$ and determine j_0 such that $\forall j \geq j_0$ $x_j \in nD$; it follows that $|\lambda_j| \leq M$ for $j \geq j_0$ and some $M > 0$. Extract a subnet $\{\lambda_{j_k}\}$ converging to λ . By the equivalence of (i) and (ii), $\text{Gr } \sigma$ is closed. Therefore $\lambda \in \sigma(x_0) \subset G$, a contradiction.

COROLLARY 2. *If any of the conditions of Theorem 1 is verified, $\sigma(x)$ is compact for every x in X , and σ maps compact sets into compact sets.*

PROOF. Since σ has a closed graph, it carries compact sets into closed sets. Moreover the image of a compact set is bounded since σ is uniformly bounded on a balanced neighborhood of zero in X . \square

THEOREM 3. *The following conditions are equivalent:*

- (i) W is a G_δ .
- (ii) $\text{Gr } \sigma$ is an F_σ .
- (iii) $\sigma^{-1}(F)$ is an F_σ in X for every closed subset F of \mathbf{C} .

If X is an F -space with continuous multiplication, the next condition is also equivalent to the above three:

- (iv) *the mapping $x \rightarrow x^{-1}$ is continuous.*

The proof of Theorem 3 is similar to the proof of Theorem 5.

The equivalence of (iv) and (i) when X is an F -space is due to Banach [3].

REMARK 4. Let $\mathbf{C}(t)$ be the quotient field of the algebra of complex polynomials in t . It is possible to define a metrizable locally convex topology on $\mathbf{C}(t)$ such that multiplication is jointly continuous and the set of invertible elements is open but such that the mapping $x \rightarrow x^{-1}$ is not continuous. For the details see [2, pp. 277, 279].

This shows that the completeness assumption in Theorem 3 is essential.

As for Borel sets of the second class, we have

THEOREM 5. *Suppose that the topology of X is completely metrizable. Then the following conditions are equivalent:*

- (i) W is an $F_{\sigma\delta}$.
- (ii) $\text{Gr } \sigma$ is a $G_{\delta\sigma}$.
- (iii) $\sigma^{-1}(F)$ is a $G_{\delta\sigma}$ for every closed subset F of the plane.

PROOF. Assume (ii). Thus $\text{Gr } \sigma = \bigcup_m G_m$, where each G_m is a G_δ in $X \times \mathbf{C}$. Let (K_n) be a sequence of compact subsets of \mathbf{C} such that $\mathbf{C} = \bigcup_n K_n$. Then

$$\sigma^{-1}(F) = \Pi_X[(X \times F) \cap \text{Gr } \sigma] = \bigcup_n \bigcup_m \Pi_X[(X \times (F \cap K_n)) \cap G_m].$$

Note that $(X \times (F \cap K_n)) \cap G_m$ is a G_δ in $X \times \mathbf{C}$ and is therefore completely metrizable. The mapping $\Pi_X|_{X \times K_n}$ is closed and continuous.

We use now the following result of Vainstein [8, p. 320, Theorem 6]: if f is a continuous closed mapping of a complete metric space Z onto a metrizable space W , then W is completely metrizable (and so an absolute G_δ). It follows that $\Pi_X[(X \times (F \cap K_n)) \cap G_m]$ is a G_δ in X .

The formula for $\sigma^{-1}(F)$ now concludes the proof.

We will show below that if X is a Polish algebra with continuous multiplication, the above conditions are fulfilled.

LEMMA 6. *Suppose that X is a Baire space and W is a G_δ . Then σ is lower semicontinuous on a dense G_δ in X .*

PROOF. By Theorem 3, $\sigma^{-1}(F)$ is an F_σ in X for each closed subset F of the plane. Since open subsets of \mathbf{C} are F_σ 's, their preimages under σ are F_σ 's too.

The mapping σ is not lsc at $x_0 \in X$ if there exists an open $V \subset \mathbf{C}$ such that $V \cap \sigma(x_0) \neq \emptyset$ and $x_0 \notin \text{int } \sigma^{-1}(V)$.

Let $K = \{x \in X: \sigma \text{ is not lsc at } x\}$. Then

$$K = \bigcup \{[\sigma^{-1}(V) \setminus \text{int } \sigma^{-1}(V)]: V \text{ is open in } \mathbf{C}\}.$$

Each set $\sigma^{-1}(V) \setminus \text{int } \sigma^{-1}(V)$ is an F_σ of the first category, and since C has a countable base the same conclusion holds for K . The lemma now follows because X is a Baire space. \square

As a consequence of Lemma 6 and Theorem 1, we have

THEOREM 7. *Suppose X is a Baire space and W is open. Then σ is continuous on a dense G_δ .*

REMARK 8. (i) By Corollary 2, $\sigma(x)$ is compact for each x in X .

Thus σ is lsc and usc if and only if it is continuous, as a point-map, with respect to the Hausdorff metric in the plane.

(ii) The hypotheses of Theorem 7 are verified by Banach algebras. We have, as a consequence, that the spectral radius is continuous on a dense G_δ . For examples of spectral discontinuities in Banach algebras see [1].

COROLLARY 9. *Suppose X is a Banach algebra. If the set M of points of continuity of the spectrum is closed under addition, it coincides with the whole algebra.*

PROOF. Since the origin belongs to M , the assumption implies that M is a vector subspace. By Theorem 7, M is topologically complete and, hence, by a result of Klee [4], it is complete. Therefore $M = X$. \square

Let Z be a Polish space and X_0 a subset of Z .

DEFINITION 10 [7]. A family $(E_n^j: j, n \in \mathbf{N})$ of subsets of X_0 is a regular system for X_0 if the following conditions hold:

- (i) each E_n^j is relatively closed in X_0 ,
- (ii) $E_n^j \subset E_n^{j+1} \forall n, j \in \mathbf{N}$,
- (iii) $\bigcup_j E_n^j = X_0 \forall n \in \mathbf{N}$,
- (iv) for each $\sigma = (\sigma_n) \in \mathbf{N}^{\mathbf{N}}$, $E(\sigma) = \bigcap_n E_n^{\sigma_n}$ is closed in Z .

In [6] it is proved that a subset of Z which admits a regular system is a Borel subset of Z .

Using the results of [5] we can prove

THEOREM 11. *A subset of Z which admits a regular system is $F_{\sigma\delta}$ in Z .*

For the proof we need

LEMMA 12. *Let Y be a metric space and suppose that $B: \mathbf{N}^{\mathbf{N}} \rightarrow 2^Y$ is a set-valued mapping satisfying the following conditions:*

- (i) $\forall \sigma \in \mathbf{N}^{\mathbf{N}}$, $B(\sigma)$ is closed in Y ,
- (ii) if $\sigma \leq \tau$ in $\mathbf{N}^{\mathbf{N}}$ (that is $\sigma_n \leq \tau_n \forall n \in \mathbf{N}$), $B(\sigma) \subseteq B(\tau)$,
- (iii) if $\sigma^1 \geq \sigma^2 \geq \dots$ and $\sigma^k \rightarrow \sigma$ in $\mathbf{N}^{\mathbf{N}}$, $B(\sigma) = \bigcap_k B(\sigma^k)$, then the graph of B is closed in $\mathbf{N}^{\mathbf{N}} \times Y$.

PROOF. Let $\sigma^k \rightarrow \sigma$ and $y_k \in B(\sigma^k)$ with $y_k \rightarrow y$ in Y . We must show that $y \in B(\sigma)$.

Define $\tilde{\sigma}^k \in \mathbb{N}^{\mathbb{N}}$ for every $k = 1, 2, \dots$ by putting

$$\tilde{\sigma}_n^k = \sup \{ \sigma_n^m : m \geq k \}.$$

Clearly $\tilde{\sigma}^k \geq \tilde{\sigma}^{k+1}$ for every k , and $\lim \tilde{\sigma}^k = \sigma$. Therefore $B(\tilde{\sigma}^{k+1}) \subset B(\tilde{\sigma}^k)$ for every k and

$$B(\sigma) = \bigcap_{k \geq 1} B(\tilde{\sigma}^k).$$

On the other hand, $\sigma^m \leq \tilde{\sigma}^m \leq \tilde{\sigma}^k$ whenever $m \geq k$, and so $B(\sigma^m) \subset B(\tilde{\sigma}^k)$ for $m \geq k$. This implies that for each k the closed set $B(\tilde{\sigma}^k)$ contains the sequence $(y_m)_{m \geq k}$ together with its limit y . Therefore $y \in B(\sigma)$.

PROOF OF THEOREM 11. Let $E: \mathbb{N}^{\mathbb{N}} \rightarrow 2^Z$ be the set-valued mapping $E(\sigma) = \bigcap_n E_n^{\sigma_n}$, where $\sigma = (\sigma_n)_n \in \mathbb{N}^{\mathbb{N}}$. $\text{Gr} E$ is closed in $\mathbb{N}^{\mathbb{N}} \times Z$ as a consequence of Lemma 12. Thus $\text{Gr} E$ is a Polish space. Let us put $\pi = \pi_{Z|\text{Gr} E}$. π is continuous and $\pi(\text{Gr} E) = X_0$, by property (iii) of regular systems and the equality $\bigcup_{\sigma} \bigcap_n E_n^{\sigma_n} = \bigcap_n \bigcup_{\sigma} E_n^{\sigma_n}$.

Let us show that π maps open subsets of $\text{Gr} E$ onto F_{σ} sets.

Let $M = \{ \sigma \in \mathbb{N}^{\mathbb{N}} : \sigma_1 = c_1, \dots, \sigma_k = c_k \}$ be a basic open set in $\mathbb{N}^{\mathbb{N}}$ and let V be open in Z . It is enough to show that $\pi[(M \times V) \cap \text{Gr} E]$ is F_{σ} . But

$$(M \times V) \cap \text{Gr} E = \{ (\sigma, x) : x \in E(\sigma) \cap V; \sigma_1 = c_1, \dots, \sigma_k = c_k \}$$

and

$$\begin{aligned} \pi[(M \times V) \cap \text{Gr} E] &= \bigcup_{\sigma \in M} (E(\sigma) \cap V) = \bigcup_{\sigma \in M} \left(\bigcap_n E_n^{\sigma_n} \cap V \right) \\ &= V \cap \left(\bigcup_{\sigma \in M} \bigcap_n E_n^{\sigma_n} \right) = V \cap \left(\bigcap_n \bigcup_{\sigma \in M} E_n^{\sigma_n} \right) \\ &= V \cap E_1^{c_1} \cap E_2^{c_2} \cap \dots \cap E_k^{c_k} \end{aligned}$$

by property (ii) of regular systems.

This shows that $\pi[(M \times V) \cap \text{Gr} E]$ is F_{σ} in X_0 , since each $E_k^{c_k}$ is closed in X_0 . By Theorem 2 of [5] we can conclude that X_0 is $F_{\sigma\delta}$ in X . \square

As an immediate corollary we have

THEOREM 13. *Let T be a continuous linear operator on an F -space. Then the spectrum of T is $G_{\delta\sigma}$ in the plane.*

PROOF. As shown in [7], the resolvent of T admits a regular system. Thus, by the preceding theorem, it is $F_{\sigma\delta}$ in \mathbb{C} .

Theorem 13 sharpens the conclusions of [7], where $\sigma(T)$ was shown to be a Borel subset of the plane and proved a $G_{\delta\sigma}$ only in the separable or locally convex case.

The preceding result naturally leads to the following open question: is every $G_{\delta\sigma}$ of the plane the spectrum of a continuous linear operator on an F -space?

THEOREM 14. *Let X be a Polish algebra with continuous multiplication. Then W is $F_{\sigma\delta}$ in X , and the mapping $x \rightarrow x^{-1}$ is a Borel measurable function of the second class.*

PROOF. Let d be a complete invariant metric on X , and for each k, n in N let

$$E_n^k = \{a \in X: \forall x \in X, d(0, x) > 1/2^n \Rightarrow d(0, ax) \geq 1/2^k \text{ and} \\ d(0, xa) \geq 1/2^k \text{ and } a \cdot B_k \subset \bar{B}_n \text{ and } B_k \cdot a \subset \bar{B}_n\},$$

where

$$B_k = \{x \in X: d(0, x) < 1/2^k\}, \\ \bar{B}_n = \{x \in X: d(0, x) \leq 1/2^n\};$$

put

$$F_n^k = \{a \in W: a^{-1} \cdot B_k \subset \bar{B}_n, B_k \cdot a^{-1} \subset \bar{B}_n, a \cdot B_k \subset \bar{B}_n, B_k \cdot a \subset \bar{B}_n\}.$$

It is easy to check that $F_n^k = W \cap E_n^k$. Each E_n^k is closed because multiplication is continuous. The family $(F_n^k: k, n \in \mathbf{N})$ is a regular system for W : conditions (i)–(iii) have straightforward verifications. Let us prove condition (iv).

Let (k_n) be a sequence of natural numbers. We remark that $(a^{-1}: a \in \bigcap_n F_n^{k_n})$ is an equicontinuous family—both on the right and on the left—of linear operators on X .

Thus, if (a_s) is a Cauchy sequence in $\bigcap_n F_n^{k_n}$, the formula

$$a_{s'}^{-1} - a_{s''}^{-1} = a_{s''}^{-1}(a_{s''} - a_{s'})a_{s'}^{-1}$$

shows that (a_s^{-1}) is also a Cauchy sequence. Let a_0 and b_0 be the limits in X of (a_s) and (a_s^{-1}) respectively. Then, by the continuity of multiplication, $a_0 b_0 = b_0 a_0 = e$ and $a_0 \in \bigcap_n F_n^{k_n}$. This proves that $\bigcap_n F_n^{k_n}$ is closed in X and $(F_n^k: n, k \in \mathbf{N})$ is a regular system for W . By Theorem 11, W is an $F_{\sigma\delta}$ in X .

Let us now put $f(x) = x^{-1}$ for x in W . It is clear that $f(F_n^k) = F_n^k$ for each k and n in N . The formulae $a - b = b(b^{-1} - a^{-1})a$ and $a^{-1} - b^{-1} = b^{-1}(b - a)a^{-1}$ ($a, b \in W$) show that for each sequence $(\sigma_n) = \sigma$ the mapping $f|_{\bigcap_n F_n^{\sigma_n}: \bigcap_n F_n^{\sigma_n} \rightarrow \bigcap_n F_n^{\sigma_n}}$ is a uniform isomorphism. Let F be closed in W ; then $(F \cap F_n^k: k, n \in \mathbf{N})$ is a regular system for F . Moreover $f(F \cap \bigcap_n F_n^{\sigma_n}) = f(F) \cap \bigcap_n F_n^{\sigma_n}$ is closed in X as $F \cap \bigcap_n F_n^{\sigma_n}$ is complete.

Therefore the mapping $B: \mathbf{N}^N \rightarrow X$ defined by $B(\sigma) = f(F \cap \bigcap_n F_n^{\sigma_n})$ satisfies the hypotheses of Lemma 12.

Since $f(F) = \Pi_X(\text{Gr} B)$ and $\Pi_{X|_{\text{Gr} B}}$ is open to F_σ on $\text{Gr} B$, it follows that $f(F)$ is an $F_{\sigma\delta}$ in X and that f , being its own inverse, is a Borel function of the second class.

□

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