LU-FACTORIZATION OF OPERATORS ON $l_1$

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Abstract. Necessary and sufficient conditions are obtained for LU-factorization of operators on $l_1$. In particular it is shown that uniform invertibility of the compressions of the operator is not sufficient to insure an LU-factorization of the operator, thus answering a question of de Boor, Jia, and Pinkus.

The question of when a bounded linear operator on $l_p$, $1 \leq p \leq \infty$, has an $LU$-factorization has been much studied recently. Barkar and Gohberg [2] have shown that if $A$ is an operator on $l_p$ which has an $LU$-factorization, then $A$ and its compressions $A_n = P_nAP_n$ are uniformly invertible, i.e. sup$_n (\|A_n^{-1}\|, \|A^{-1}\|) < \infty$. In the other direction, various classes of operators such as invertible, diagonally dominant operators on $l_1$ [7] and invertible, totally positive operators [3, 1] on $l_p$ have been shown to have $LU$-factorizations. For these kinds of operators it is known [1] that their compressions satisfy a stronger condition than uniform invertibility; namely, that the inverses of the compressions are order bounded, i.e. $\|\text{sup}_n A_n^{-1}\| < \infty$. Left open, then, is the possibility (first raised in [3] with a negative expectation) that uniform invertibility might be sufficient for a matrix operator on $l_\infty$ to have an $LU$-factorization. In this paper an example is given that shows that uniform invertibility is not sufficient for factoring an operator on $l_\infty$ (or $l_1$). However, we also show that uniform invertibility of the compressions is sufficient to ensure an $LU$-factorization when the operator has an inverse whose columns decay at a certain rate away from the diagonal. Among the operators with this property are the banded operators.

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We now fix some terminology and notation. If $x = (x_i)$ is an element of $l_1$ we denote its usual projection onto the span of the first $n$ basis vectors by $P_nx$. A bounded linear operator $A$ on $l_1$ is said to be upper (respectively lower) triangular if $P_nAP_n = AP_n$ (respectively $P_nA$) for all $n$. We say that $A$ is unit upper (lower) triangular if it is upper (lower) triangular and its diagonal entries in the matrix representation for $A$ relative to the usual basis $e_i$ of $l_1$ are all ones. An operator $A$ is said to have an $LU$-factorization (relative to the usual basis $e_i$ of $l_1$) if there exist invertible operators $L$ and $U$ so that $A = LU$ and the operators $L$, $L^{-1}$ are unit lower triangular while $U$, $U^{-1}$ are upper triangular. An operator $A$ is said to be

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banded if there exist integers $m$ and $l$ so that $A(i, j) = 0$ if $j \not\in [i - l, i - l + m]$. The absolute value of an operator $A = (a_{ij})$ is the operator $|A| = (|a_{ij}|)$. Finally, we let $A_n^{-1}$ denote the operator on $l_1$ whose decomposition with respect to $P_n$ and $I - P_n$ is given by

$$
\begin{pmatrix}
(P_n A P_n)^{-1} & 0 \\
0 & 0
\end{pmatrix}.
$$

**Example.** For each $m$, let $B_m$ be the operator on $l_1^m$ given by $B_m e_j = e_1 - e_{j+1}$, $j = 1, 2, \ldots, m - 1$, and $B_m e_m = e_1$. Then each $B_m$ is invertible relative to $l_1^m$, in fact, $B_m^{-1} e_1 = e_m$ and $B_m^{-1} e_j = e_m - e_{j-1}$, $j = 2, 3, \ldots, m$. Since for each $i$, $P_i B_m P_i = B_i$, we have that the compressions of each $B_m$ are invertible and so each $B_m$ has an $LU$-factorization [4, p. 178]. In fact, $B_m = L_m U_m$ where $L_m e_j = e_j - e_{j+1}$, $j = 1, 2, \ldots, m - 1$, and $L_m e_m = e_m$ and $U_m e_j = \sum_{k=1}^{j} e_k$, $j = 1, 2, \ldots, m$. Note that $\|U_m\| = m$. If we now let $A = \bigoplus_{m=1}^{\infty} B_m$ then $A$ and its compressions are uniformly invertible; in fact, sup $\|A_n^{-1}\|$, $\|A^{-1}\|$, $\|A\|$ = 2. But if $A = LU$ then $\|U\| \geq$ sup $\|P_n U P_n\|$ $\geq$ sup $\|U_m\|$ = $\infty$, so $A$ does not have an $LU$-factorization. This fact can also be easily obtained using Theorem 2 of [1] since $B_m^{-1} e_1 = e_m$ implies that $(\sup_m B_m^{-1}) e_1 = \sum_m e_m$, i.e. $\sup \|B_m^{-1}\| = \infty$. Consequently, the block diagonal matrix $A$ must also have $\|\sup_{n} A_{n}^{-1}\| = \infty$ and so does not have an $LU$-factorization. We remark that $A^* : l_\infty \to l_\infty$ does not have an $LU$-factorization either. For if $A^* = LU$, since $L$ and $U$ are operators on $l_\infty$ representable as matrices, $A = U_* L_*$ is an $LU$-factorization for $A$ where $U_*$ and $L_*$ are the preadjoints of $U$ and $L$ [8]. This fulfills the expectation raised in [3].

The question remains as to whether there are any easily recognized situations in which uniform invertibility of the compressions is sufficient to insure an $LU$-factorization of the operator. In order to give an example of such a situation we find it convenient to give a characterization of when an operator on $l_1$ has an $LU$-factorization. This characterization is similar to that presented in Theorem 2 of [1] where the finiteness of $\|\sum A_{n+1}^{-1} - A_{n+1}^{-1}\|$ is replaced by the finiteness of $\|\sup_n A_{n}^{-1}\|$. As further motivation we recall that if an operator $A$ and its compressions are uniformly invertible, then $A_n^{-1} e_i \to A^{-1} e_i$ for all $i$. Our first result shows that for $A$ to have an $LU$-factorization this convergence must be of a telescoping variety.

**Theorem 1.** A bounded linear operator $A$ on $l_1$ has an $LU$-factorization if and only if, for each $n$, $A_n = P_n A P_n$ is invertible and

$$
\sup_i \sum_{n=1}^{\infty} \| (A_{n+1}^{-1} - A_{n+1}^{-1}) e_i \| = \left\| \sum_{n=1}^{\infty} (A_{n+1}^{-1} - A_{n+1}^{-1}) \right\| < \infty.
$$

**Proof.** If $A = LU$ then $A_n = P_n L P_n U P_n$ and hence $A_n^{-1} = P_n U^{-1} P_n L^{-1} P_n = U^{-1} U_i L^{-1}$ since $U^{-1}$ is upper triangular and $L^{-1}$ is lower triangular. Consequently, $(A_{n+1}^{-1} - A_{n+1}^{-1})(e_i) = U^{-1}(P_{n+1} - P_n)L^{-1} e_i$, so

$$
\sup_i \sum_{n=1}^{\infty} \| (A_{n+1}^{-1} - A_{n+1}^{-1}) (e_i) \| \leq \sup_i \| U^{-1} \| \sum_{n=1}^{\infty} \| (P_{n+1} - P_n) L^{-1} e_i \| \leq \| U^{-1} \| \sup_i \| L^{-1} e_i \| = \| U^{-1} \| \| L^{-1} \| < \infty.
$$
For the converse, note that the hypothesis implies that

\[ Be_i = A_n^{-1}e_i + \sum_{n=1}^{\infty} (A_{n+1}^{-1} - A_n^{-1})e_i \]

exists for each \( i \) and \( \sup \| Be_i \| < \infty \). Hence \( B \) extends to a bounded linear operator on \( l_1 \) and since \( Be_i = \lim_n A_n^{-1}e_i \) it follows quickly that \( B = A^{-1} \). Now for each \( N \),

\[ A_n^{-1}e_i = A_N^{-1}e_i + \sum_{n=N}^{\infty} (A_{n+1}^{-1} - A_n^{-1})e_i \]

and so

\[ A_N^{-1} = A^{-1} + \sum_{n=N}^{\infty} (A_{n+1}^{-1} - A_n^{-1}) \]

pointwise. Hence

\[ \sup_N |A_N^{-1}| \leq |A^{-1}| + \sum_n |A_{n+1}^{-1} - A_n^{-1}| \]

pointwise and, consequently,

\[ \| \sup_N |A_N^{-1}| \| \leq \| A^{-1} \| + \| \sum_n |A_{n+1}^{-1} - A_n^{-1}| \| < \infty. \]

Now since \( A_n \) is invertible for all \( n \), we have that \( A_n = L_nU_n \). We shall show that the operators \( L_n^{-1} \) and \( U_n^{-1} \) are bounded and so deduce that \( A \) has an \( LU \)-factorization. (This part of the argument has already appeared in [1] but we include it here for the sake of completeness.) Now for each \( n \),

\[ L_n^{-1}(i, j) = - \sum_{k=1}^{i-1} A_{i-1}^{-1}(k, j) A(i, k) \quad \text{for } i > j \]

and

\[ U_n^{-1}(i, j) = A_j^{-1}(i, j) \quad \text{for } i < j \]

[1, 2]. It follows that

\[ \sup_n |L_n^{-1}(i, j)| \leq \sum_{k=1}^{\infty} \sup_i |A_{i-1}^{-1}(k, j)||A(i, k)| \quad \text{for } i > j \]

and so

\[ \sup_n \| L_n^{-1} \| \leq \| \sup_i |L_n^{-1}| \| \leq \| \sup_i |A_{i-1}^{-1}| \||A| + 1 < \infty. \]

Similarly,

\[ \sup_i \| U_n^{-1} \| \leq \| \sup_n |U_n^{-1}| \| \leq \| \sup_n |A_n^{-1}| \| < \infty. \]

Since \( L_n = P_nL_{n+1}P_n \) and \( U_n = P_nU_{n+1}P_n \) we have that \( L_n^{-1} = P_nL_{n+1}^{-1}P_n \) and \( U_n^{-1} = P_nU_{n+1}^{-1}P_n \). Consequently, for each \( x \) in \( l_1 \), the limits \( \lim_n L_nx = Lx \), \( \lim_n L_n^{-1}x = Vx \), \( \lim_n U_nx = Ux \), and \( \lim_n U_n^{-1}x = Wx \) exist and define bounded triangular operators on \( l_1 \). Now since

\[ LVx = \lim_n L_nL_n^{-1}x = \lim_n I_nx = x = \lim_n I_nx = \lim_n L_n^{-1}L_nx = VLx \]
we have that $V = L^{-1}$. Similarly, $W = U^{-1}$. Finally, for each $x$ in $l_1$, we have that $LUX = \lim_n L_n U_n x = \lim_n A_n x = Ax$ so $A$ has the promised factorization.

We remark that Theorem 1 can be easily applied to the example preceding the theorem. In this case

$$A_2^{-1} - A_1^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_3^{-1} - A_2^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$A_4^{-1} - A_3^{-1} = \begin{pmatrix} \vdots & \cdots & 0 \\ \vdots & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_5^{-1} - A_4^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$A_6^{-1} - A_5^{-1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

(Here we have displayed only the upper left hand, nonzero portion of each operator.) Hence

$$\sum_{n=1}^{\infty} \left\| (A_{n+1}^{-1} - A_n^{-1}) (e_2) \right\| = 3, \quad \sum_{n=1}^{\infty} \left\| (A_{n+1}^{-1} - A_n^{-1}) (e_4) \right\| = 5$$

and, in general, by a routine but tedious induction argument,

$$\sum_{n=1}^{\infty} \left\| (A_{n+1}^{-1} - A_n^{-1}) (e_{k(k+1)/2+1}) \right\| = 2k + 1$$

so by Theorem 1 $A$ does not have an $LU$-factorization.

**Theorem 2.** Let $A$ be a bounded linear operator on $l_1$. If $A$ and its compressions are uniformly invertible and, in addition,

$$\sup_i \sum_{k=1}^{\infty} k \left| A^{-1}(i, i) \right| < \infty,$$

then $A$ has an $LU$-factorization.

**Proof.** We start with $AA^{-1} e_i = e_i$. Hence

$$AP_n A^{-1} e_i + A(I - P_n) A^{-1} e_i = e_i \quad \text{for all } n$$

so

$$P_n AP_n A^{-1} e_i + P_n (I - P_n) A^{-1} e_i = P_n e_i = e_i \quad \text{for } i \leq n.$$ Hence

$$P_n A^{-1} e_i - A_n^{-1} e_i = A_n^{-1} P_n A (P_n - 1) A^{-1} e_i \quad \text{for } i \leq n$$

and thus

$$\left\| P_n A^{-1} e_i - A_n^{-1} e_i \right\| \leq M^2 \left\| (P_n - I) A^{-1} e_i \right\| \quad \text{for } i \leq n$$

where $M = \sup_n \{ \left\| A_n^{-1} \right\|, \left\| A \right\| \}$. Now

$$\left\| A_n^{-1} e_i - A^{-1} e_i \right\| \leq \left\| P_n A^{-1} e_i - A_n^{-1} e_i \right\| + \left\| (I - P_n) A^{-1} e_i \right\|$$

$$\leq (M^2 + 1) \left\| (I - P_n) A^{-1} e_i \right\| \quad \text{for } i \leq n.$$
Hence
\[
\sum_{n=i}^{\infty} \| A_{n+1}^{-1} e_i - A_n^{-1} e_i \| \leq 2(M^2 + 1) \sum_{n=i}^{\infty} \| (I - P_n) A_n^{-1} e_i \|
\]
\[
\leq 2(M^2 + 1) \sum_{n=i}^{\infty} \sum_{j=n+1}^{\infty} | (A^{-1} e_i, e_j) |
\]
\[
\leq 2(M^2 + 1) \sum_{k=1}^{\infty} k | (A^{-1} e_i, e_{i+k}) |.
\]

Consequently,
\[
\sup_{n=1}^{\infty} \sum_{i} \| (A_{n+1}^{-1} - A_n^{-1}) e_i \| \leq \sup_i \| A_i^{-1} e_i \| + \sup_i \sum_{n=i}^{\infty} \| (A_{n+1}^{-1} - A_n^{-1}) (e_i) \| < \infty.
\]

So by Theorem 1 A has an LU-factorization.

It is not surprising that the additional condition imposed in Theorem 2 is far from necessary. For example, choose a so that 0 < a < 1 and \( \sum_n (a/n^2) < 1 \) and let
\[
B(i, j) = \begin{cases} a/(i - 1)^2, & j = 1, i \neq 1, \\ 0, & \text{otherwise}. \end{cases}
\]

Then \( \| B \| < 1 \), and \( B(i, i) = 0 \) for all \( i \) so \( A = I - B \) is invertible and strictly (column) diagonally dominant. Consequently, A has an LU-factorization [7]. But since \( B^2 = 0 \), \( A^{-1} = I + B \) and so
\[
\sum_{k=1}^{\infty} k | A^{-1}(k + 1, 1) | = \sum_{k=1}^{\infty} k \frac{a}{k^2} = \infty.
\]

The problem here is the slowness of the decay rate of the entries of \( A^{-1} \) away from the main diagonal; however, for banded operators this poses no difficulty.

**Corollary 3.** Let A be a banded operator on \( l_1 \). Then A has an LU-factorization if and only if A and its compressions are uniformly invertible.

**Proof.** One direction is clear; for the other we recall from [5] that if A is banded and invertible then there are positive constants C and \( \lambda \) with \( \lambda < 1 \) so that
\[
| A^{-1}(i, j) | \leq C \lambda^{i-j} \text{ for all } i, j.
\]

Consequently,
\[
\sup_i \sum_{k=1}^{\infty} k | A^{-1}(i + k, i) | \leq C \sum_{k=1}^{\infty} k \lambda^k < \infty.
\]

So Theorem 2 shows that A has an LU-factorization.

**References**


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