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FOUR COUNTEREXAMPLES TO BLOCH'S PRINCIPLE

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Dedicated to the memory of Loo-Keng Hua

ABSTRACT. In this note, four counterexamples are given to Bloch's heuristic principle in complex function theory. The first involves univalent functions, the second certain autonomous differential equations, and the remaining two involve certain autonomous differential expressions omitting certain values.

One version of the "Bloch principle" (everyone calls it that, but no one seems to know why) asserts that if we consider a property $\mathscr{P}$ of analytic functions such that every *entire* function with that property must be constant, then the class of analytic functions in the unit disk **D** with that property forms a normal family. The most cited instance of this heuristic principle is where we are given two distinct complex numbers w_1 and w_2, and we take $\mathscr{P} = \mathscr{P}_{w_1,w_2}$ to be the property that the analytic function f omit w_1 and w_2. In this instance, the Bloch principle is correct. Of course, if we took $\mathscr{P}$ to be the property that f omit two distinct values $w_1(f)$ and $w_2(f)$ (that depend on f), then we would have a "counterexample". In this paper, we provide four counterexamples that are considerably more subtle.

In **[ZAL-I]**, Zalcman proved a precise formulation (due essentially to Abraham Robinson—see **[ROB]**) of the Bloch principle. This formulation is useful in many circumstances, but the hypotheses prevent it from being used in other circumstances, even though the Bloch principle is valid in those cases. In my experience, the hypothesis of **[ZAL-I]** that is most often not met is that of linear invariance—that if $f(z)$ has property $\mathscr{P}$ on its domain, then $f(az + b)$ has property $\mathscr{P}$ on *its* domain, whatever the complex numbers $a \neq 0$ and b may be. For simplicity, we restrict ourselves here to the case where f is analytic, but most other authors consider the more general case of meromorphic functions.

COUNTEREXAMPLE 1. Let $\mathscr{P}$ be the property of the analytic function f that

(1) $f = g''$ for some univalent function g, i.e. that f is the second derivative of a univalent function.

(A) If f is an entire function for which $\mathscr{P}(f)$ holds, then f is constant (actually $f = 0$). For then g is a univalent entire function, so that, as is very well known, $g(z) = Az + B$ for some constants A and B, and hence $f = g'' = 0$.

(B) Consider in **D** the functions $g_n(z) = n(z + z^2/100 + z^3/100)$, for $n = 1, 2, 3, \ldots$. Then $g_n'(z) = n(1 + 2z/100 + 3z^2/100)$. Clearly $\operatorname{Re} g_n'(z) > 0$ in **D**, so that by the (easily proved) Noshiro-Warschawski Theorem (that any function on a

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convex domain, whose derivative has strictly positive real part, must be univalent), each g_n is univalent in $\mathbf{D}$. Now let $f_n(z) = g_n''(z) = n(2/100 + 6z/100)$. For $z = -2/6$, $f_n(z) = 0$, while for $z \neq -2/6$, $f_n(z) \to \infty$ as $n \to \infty$. Hence $\{f_n(z)\}$ is *not* a normal family, and the Bloch principle does not hold in this case.

REMARK. If, instead, we had taken $\mathscr{P}$ as the property that either $f = 0$ or $f = g'$ for some univalent function g, then this $\mathscr{P}$ *does* satisfy the hypotheses of **[ZAL-I]**, as is easily checked. We see then that the class, in $\mathbf{D}$ (or any region, for that matter) of *first* derivatives of univalent functions forms a normal family. It is amusing to note that one may now apply the Marty criterion for normality,

$$|g''(z)|/\left(1 + |g'(z)|^2\right) \leqslant M(z),$$

where $M(z)$ is some positive continuous function on $\mathbf{D}$, focussing on the class $\mathscr{S}$ of normalized univalent functions $g(z) = z + a_2 z^2 + \cdots$ to conclude that $|a_2| \leqslant M$, where M is independent of g. Hardly a sharp result, but the chain of the argument is interesting, resting eventually on the Casorati-Weierstrass Theorem.

COUNTEREXAMPLE 2. Consider the simultaneous autonomous algebraic differential equations

$$y'^2 + 4y^2 - 4 = 0, \tag{2}$$

$$y'' + 4y = 0. \tag{3}$$

The solutions are exactly $y = \sin 2(z - a)$, for $a \in \mathbf{C}$. (Note that the purpose of (3) is to exclude the singular solutions $y = \pm 1$ of (2).) Now write, formally,

$$y = 2f(z)/f'(z) \tag{4}$$

and put this into (2) and (3), expand, and clear the denominators to get

$$P(f(z), f'(z), f''(z)) = 0, \tag{2'}$$

$$Q(f(z), f'(z), f''(z), f'''(z)) = 0 \tag{3'}$$

where P and Q are polynomials with (real) constants as coefficients. It would only take a few minutes to write out P and Q explicitly. Now let $\mathscr{P}(f)$ be the property that the analytic function f satisfy (2′) and (3′) on its domain.

(A) First, I claim that the only *entire* functions f with property $\mathscr{P}$ are constants. For if f were not a constant, then we would have to have $f(z) = k\tan(z - a)$ for some a, $k \in \mathbf{C}$. To see this, go to (2) and (3), remembering that $y = 2f(z)/f'(z)$, to get

$$\frac{f'(z)}{f(z)} = \frac{1}{\sin(z - a)\cos(z - a)}.$$

(Since $f \neq$ const., the division by $f'(z)$ is alright.) But

$$\int [\sin w \cos w]^{-1}\, dw = \log \tan w,$$

and the result follows. Now if $f(z) = k\tan(z - a)$ is entire, then we must have $k = 0$, so that $f \equiv 0$ is a constant after all.

(B) However, in $\mathbf{D}$, the functions $f_n(z) = n \tan z$ all have property $\mathscr{P}$ (easily checked, as in part (A)), but $\{f_n(z)\}$ is not a normal family, since, again, $f_n(0) = 0$ for all n while $f_n(z) \to \infty$ as $n \to \infty$ for all $z \in \mathbf{D} \setminus \{0\}$. This is another failure of the Bloch principle.

COUNTEREXAMPLE 3. Let P be the autonomous (i.e. constant-coefficient) differential polynomial

$$P(\vec{f}) = (f'(z) - 1)(f'(z) - 2)(f'(z) - f(z)),$$

and let $\mathscr{P}$ be the property that $P(\vec{f})(z)$ omits the value 0. Then (A) every entire function with property $\mathscr{P}$ must be constant, yet (B) the class of analytic functions in the unit disc, with property $\mathscr{P}$ does not form a normal family.

PROOF. (A) Suppose the entire function f has property $\mathscr{P}$. By Picard's small theorem, $f'(z) = c$, a constant, so $f(z) = cz + d$. But since $f'(z) - f(z)$ omits 0, we must have $c = 0$.

(B) In $\mathbf{D}$, let $f_n(z) = nz$, $n = 5, 6, 7, \ldots$. Then $f_n'(z) = n \neq 1, 2$. And since $n - nz$ is zero-free in $\mathbf{D}$, $f_n'(z) - f_n(z)$ omits 0. Thus, each f_n has property $\mathscr{P}$ in $\mathbf{D}$. Yet $\{f_n\}$ is not a normal family since $f_n(0) = 0$ while $f_n(z) \to \infty$ as $n \to \infty$ for each $z \in \mathbf{D} \setminus \{0\}$.

For a while, I thought that if one would insist that f omit zero, in addition to a "differential omitting" as in the previous example, i.e., that $P(\vec{f})$ have f as a factor, then Bloch's principle would hold, but the next counterexample scotches this hope.

COUNTEREXAMPLE 4. Let

$$P(\vec{f}) = ff'[ff'' - f'^2][ff'' - f'^2 - f^2],$$

and, as above, let $\mathscr{P}$ be the property that $P(\vec{f})(z)$ omits 0. Then (A) and (B), as above, are true.

PROOF. (A) Suppose $\mathscr{P}(f)$ holds for f entire. Since f omits 0, then $f = \exp g$ for some entire function g. Then $f' = e^g g'$, $f'' = e^g[g'^2 + g'']$, and so

$$(ff'' - f'^2)/f^2 = g''.$$

Because $\mathscr{P}(f)$ holds, we see that g'' omits the two values 0 and 1, so that $g'' = c(\neq 0, 1)$, and hence $g' = cz + b$. Since g' omits 0, we must have $c = 0$, a contradiction. So the class of functions f with property $\mathscr{P}$ is *empty*, and a fortiori a subset of the constant functions.

(B) In $\mathbf{D}$, take $f_n = \exp g_n$ where $g_n(z) = nz + z^2/100$. Thus $g_n'(z) = n + z/50$, $g_n''(z) = 1/50$. Clearly, g_n' omits 0 in $\mathbf{D}$, and g_n'' omits 0 and 1 in $\mathbf{D}$. Hence, as in (A), each f_n has property $\mathscr{P}$. But $f_n(0) = 1$ for all n while, for all x with $0 < x < 1$, $f_n(x) \to \infty$ as $n \to \infty$. Therefore, $\{f_n\}$ is not a normal family in $\mathbf{D}$, and we have another counterexample.

Perhaps there is some precise form of the Bloch principle, expressed in terms of certain differential polynomials in f omitting certain values, but in the light of the last two counterexamples, it is hard to imagine what it would be. Such a theorem, though, would be very welcome, since numerous special "differential omittings" have been studied in this context, some successfully and others not, but always by ad hoc methods. (See [**CHU, HAY, KUY, LAN** and **YAN-I** and **YAN-II**].) The paper

[ZAL-II] contains an illuminating discussion of the Bloch principle that is also amusing to read. Finally, [MIN] considered a related heuristic principle and presented a counterexample in the meromorphic case.

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