PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 98, Number 2, October 1986

A UNIVERSAL EXHAUSTING DOMAIN

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ABSTRACT. A bounded domain $D \subset \mathbf{C}^n$ is constructed such that every domain $G \subset \mathbf{C}^n$ is a monotone union of biholomorphic images of D.

I. Introduction. It is widely known that two domains in $\mathbf{C}^n$, $n > 1$, are very rarely biholomorphically equivalent. In this paper we construct a domain $D \subset \mathbf{C}^n$, $n \geqslant 1$, that can be used to approximate any domain in $\mathbf{C}^n$ and therefore is "almost equivalent" to any domain in $\mathbf{C}^n$.

Let D, G be domains in $\mathbf{C}^n$. We will say that G can be exhausted by D if for every compact $K \subset G$ there exists a biholomorphic imbedding $F\colon D \to G$ such that $F(D) \supset K$.

Given D the question is to describe such domains G that can be exhausted by D. Related questions are discussed in [**1–5**]. If G is a complete hyperbolic manifold, the following two results are known. If D is a ball, polydisk or any bounded homogeneous domain, then there is only one choice of G, that is G is biholomorphically equivalent to D (see [**2, 4**]). If D is strictly pseudoconvex with a C^3 boundary, then G is biholomorphically equivalent to either D or to B, the unit ball in $\mathbf{C}^n$ (see [**4**]).

In this paper we are going to construct a universal exhausting domain.

THEOREM 1. *There exists a bounded domain $D \subset \mathbf{C}^n$, $n \geqslant 1$, such that every domain $G \subset \mathbf{C}^n$ can be exhausted by D.*

COROLLARY 1. *There exists a bounded domain $D \subset \mathbf{C}^n$ such that every domain $G \subset \mathbf{C}^n$ is a monotone union of biholomorphic images of D.*

This means that $G = \bigcup_{s=1}^{\infty} F_s(D)$, where $F_s\colon D \to G$ is a biholomorphic imbedding and $F_s(D) \subset F_{s+1}(D)$ for all s.

The construction of a universal domain allows us also to prove

THEOREM 2. *There exist two bounded domains D_1, D_2 in $\mathbf{C}^n$ such that each of them can be exhausted by the other but they are not biholomorphically equivalent.*

II. Construction of a universal exhausting domain. We use the following notations. If $z \in \mathbf{C}^n$, $n > 1$, then $z = (z_1, z')$, where $z' = (z_2, \ldots, z_n)$. $B(z,r) = \{w \in \mathbf{C} \mid |w - z| < r\}$; $B = B(0,1)$, the unit ball in $\mathbf{C}^n$. ∂D is the boundary of D. p, q are points on ∂B, $p = (1, 0, \ldots, 0)$, $q = -p$.

Received by the editors September 24, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 32H99.

$\operatorname{Aut}(B)$ is the group of holomorphic automorphisms of B. Information about the structure and properties of $\operatorname{Aut}(B)$ can be found in [6].

1. LEMMA 1. *For any* $\varepsilon > 0$ *and* $R > 0$ *there exist* $r > 0$ *and* $T \in \operatorname{Aut}(B)$ *such that*
(1.1) $R > r > 0$,
(1.2) $T(\overline{B \setminus B(p, R)}) \subset B(q, \varepsilon)$,
(1.3) $T(\overline{B(p, r)} \cap B) \subset B(p, \varepsilon)$.

PROOF. Consider for $0 < \lambda < 1$, $T_\lambda \in \operatorname{Aut}(B)$,

$$T_\lambda z = \left\{ \frac{z_1 - \lambda}{1 - z_1\lambda}, \sqrt{1 - \lambda^2}\, \frac{z'}{1 - z_1\lambda} \right\}.$$

One can see that for $z = (z_1, z') \in V = B \setminus B(p, R)$, $\operatorname{Re} z_1 < 1 - \nu$, where $\nu > 0$. Therefore, when $\lambda \to 1$, $T_\lambda(V) \to q$ uniformly on V. So, for given $\varepsilon > 0$ we can find λ_0, $T = T_{\lambda_0}$ such that (1.2) is satisfied. (1.1) and (1.3) can now be satisfied by choosing a small enough $r > 0$. It is possible because $T(p) = p$ and T is continuous at p.

2. LEMMA 2. *For any domain* U *which is a finite union of open balls*, $U = \bigcup_{s=1}^{N} B(z^s, r_s)$, *and any compact* $K \subset U$, *there exist an* $\varepsilon > 0$ *and* $F\colon U \to \mathbf{C}^n$, F *is holomorphic, such that*
(2.1) $F(U) \cap B$ *is connected*,
(2.2) $W(\varepsilon) \supset F(K)$, *where* $W(\varepsilon) = B \setminus (B(p, \varepsilon) \cup B(q, \varepsilon))$,
(2.3) $F(U) \supset B(p, \varepsilon) \cup B(q, \varepsilon)$,
(2.4) F^{-1} *is one-to-one on* $F(U) \cap B$.

PROOF. (1) First we take a ball of a minimal radius that contains U. Without any loss of generality we can assume that this ball is the unit ball B. One can prove now that $\partial B \cap \partial U$ contains at least two different points ζ, η.

(2) Now we find a $T \in \operatorname{Aut}(B)$ such that $T\zeta = p$, $T\eta = q$. T is analytic in a neighborhood B_0 of $\overline{B}$.

(3) We find now such a small $\delta > 0$ that if we introduce $\phi_\delta\colon \mathbf{C}^n \to \mathbf{C}^n$; $\phi_\delta\colon z \mapsto (1 + \delta)z$, then $\phi_\delta(U)$ has the following properties:
(a) $\phi_\delta(U) \subset B_0$,
(b) $\phi_\delta(U) \cap B$ is connected,
(c) $\phi_\delta(U) \cap B \supset \phi_\delta(K)$.

(4) We take now $F = T \circ \phi_\delta$.

(2.1) and (2.4) follow from the construction of F. The existence of an $\varepsilon > 0$ such that (2.2) and (2.3) are satisfied follows from the facts that $\phi_\delta(U) \ni \zeta$, η, and therefore $F(U) \ni p, q$, that $F(K) \subset B$, and that $F(U)$ is open.

3. Let as before a domain $U = \bigcup_{i=1}^{N} B(z^i, r_i)$.

We take

$$K_s = \bigcup_{i=1}^{N} \overline{B(z^i, r_i - 1/s)}\,.$$

$\{K_s\}$ is a sequence of compacts in U such that
(a) $K_s \subset K_{s+1}$ for every $s \geqslant 1$.
(b) $\bigcup_{s=1}^{\infty} K_s = U$.

(c) If K is any compact in U, then there exists an s such that $K \subset K_s$.

Let $1 > R > r > 0$. Denote, for $\zeta \in \partial B$,

$$D(R, r, \zeta) = \left[B(\zeta, r) \setminus \overline{B(\zeta, r)}\right] \cap B.$$

$0 < R < 1$ and a point $\zeta \in \partial B$ are now given. $U = \bigcup_{s=1}^{\infty} K_s$ is from above.

LEMMA 3. *There exist sequences* $\{R_s\}$ *and* $\{r_s\}$ *and closed sets* V_s, $1 \leqslant s < \infty$, *such that, for all* $s \geqslant 1$,

(3.1) $R_1 = R$,

(3.2) $R_s > r_s > R_{s+1} > 0$,

(3.3) $V_s \subset D_s$, *where* $D_s = D(R_s, r_s, \zeta)$.

(3.4) *There exists a biholomorphic imbedding* $\phi_s\colon (B \setminus V_s) \to U$ *such that*

$$\phi_s(D_s \setminus V_s) \supset K_s.$$

(3.5) $B \setminus V_s$ *is connected.*

PROOF. Using a unitary transformation (if needed), we may assume $\zeta = p$.

We will construct R_s by induction. For every R_s we construct r_s, V_s, ϕ_s and then R_{s+1}.

For $s = 1$ we take $R_1 = R$. Suppose R_s has been constructed. Using Lemma 2 we can find $F = F_s\colon U \to \mathbf{C}^n$ such that (2.1)–(2.4) hold for some $\varepsilon = \varepsilon_s > 0$ and $K = K_s$. Applying Lemma 1 now for ε_s and R_s we find $r = r_s > 0$ and $T = T_s \in \operatorname{Aut}(B)$ such that (1.1)–(1.3) hold. From (1.2), (1.3) and (2.2) we find

$$T_s(D_s) \supset W(\varepsilon_s) \supset F_s(K_s).$$

We choose now $V_s = T_s^{-1}(\overline{W(\varepsilon_s)} \setminus F_s(U))$, $\phi_s = F_s^{-1} \circ T_s$ and R_{s+1} is any positive number less than r_s.

Properties (3.1)–(3.5) can be checked now by using (2.1)–(2.4).

4. Consider the set S of all such domains U each of which is a finite union of open balls in $\mathbf{C}^n$ with centers at rational points and rational radii. Evidently, S is countable. $S = \{U_1, U_2, \dots, U_m, \dots\}$

5. We set, for $m \geqslant 1$, $\zeta_m = (\exp(\pi i/m), 0, \dots, 0) \in \partial B$. Choose now numbers $R(\zeta_m)$, $m \geqslant 1$, such that $R(\zeta_m) > 0$ and $B(\zeta_m, R(\zeta_m)) \cap B(\zeta_s, R(\zeta_s)) = \varnothing$ if $m \neq s$. One can take, say, $R(\zeta_m) = |\zeta_{m+1} - \zeta_m|/2$. For each m we use now the paragraph 3 to represent $U_m = \bigcup_{s=1}^{\infty} K_{ms}$ and then Lemma 3 (where $\zeta = \zeta_m$, $R = R(\zeta_m)$) to find V_{ms} and ϕ_{ms}. Let

$$D = B \setminus \bigcup_{m,s=1}^{\infty} V_{ms}.$$

Connectedness of D follows from the construction of all V_{ms} (each V_{ms} lies in a different open set) and (3.5). The universal property of D follows from the following. Given a domain $G \subset \mathbf{C}^n$ and a compact $K \subset G$ we can always find m, s such that $U_m \in S$, $G \supset U_m \supset K_{ms} \supset K$. From the construction and (3.4) one can conclude now that $\phi_{ms}\colon D \to U_m$ is a biholomorphic imbedding and $\phi_{ms}(D) \supset K_{ms}$. So, $G \supset \phi_{ms}(D) \supset K$.

III. 1. Proof of Theorem 1 follows from the construction of D and was presented above.

2. Proof of the Corollary 1. One can always represent a domain G as $G = \bigcup_{s=1}^{\infty} G_s$ such that, for all $s \geqslant 1$,

(a) G_s is a subdomain in G,

(b) $\overline{G}_s$ is a compact set in G,

(c) $G_{s+1} \supset \overline{G}_s$.

Now using Theorem 1 take F_s: $D \to G_{s+1}$ such that F_s is a biholomorphic imbedding and $F_s(D) \supset \overline{G}_s$. Eviently, $F_{s+1}(D) \supset F_s(D)$ and $G = \bigcup_{s=1}^{\infty} F_s(D)$.

3. Proof of Theorem 2. V_s was constructed in Lemma 3. One can see that we can require in addition to (3.1)–(3.5) the following: V_s has no isolated points. Actually if we take V_s constructed before and add sufficiently small neighborhoods of its isolated points the closure of the new set will still satisfy (3.3)–(3.5). Now, using this we see that a domain $D_1 = D$ can be constructed in such a way that its boundary does not have any isolated points, but D_1 still satisfies Theorem 1. Let $a \in D_1$, choose $D_2 = D_1 \setminus \{a\}$. D_2 will also be a universal exhausting domain.

Since D_1 and D_2 both are universal exhausting domains they are mutually exhaustable.

Now we need to prove that D_2 is not holomorphically equivalent to D_1. If it is, then let F: $D_2 \to D_1$ be a biholomorphism. F has a removable singularity at $a \in \partial D_2$. So, F can be uniquely extended to $\{a\}$ as a holomorphic map. Let $F(a) = b \in \overline{D}_1$. $b \in D_1$ since F is an open map and ∂D_1 has no isolated singularities. Let $c \in D_2$ be such a point that $F(c) = b$. Now if $W_c \cap W_a = \varnothing$ are neighborhoods of c and a, respectively, $F(W_c) \cap F(W_a)$ is not empty and open. This contradicts the suggestion that F is one-to-one on D_2.

References

1. H. Alexander, *Extremal holomorphic imbeddings between ball and polydisc*, Proc. Amer. Math. Soc. **68** (1978), 200–202.

2. J.-E. Fornaess and E. L. Stout, *Polydiscs in complex manifolds*, Math. Ann. **227** (1977), 145–153.

3. J.-E. Fornaess and Sibony Nessim, *Increasing sequences of complex manifolds*, Math. Ann. **255** (1981), 351–360.

4. B. L. Fridman, *Biholomorphic invariants of a hyperbolic manifold and some applications*, Trans. Amer. Math. Soc. **276** (1983), 685–698.

5. L. Lempert, *A note on mapping polydiscs into balls and vice versa*, Acta Math. Hungar. **34** (1979), 117–119.

6. W. Rudin, *Function theory in the unit ball of* $\mathbf{C}^n$, Springer-Verlag, New York, 1980.

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