A COMPLEX SPACE WHOSE SPECTRUM IS NOT
LOCALLY COMPACT ANYWHERE
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ABSTRACT. An example of a two-dimensional complex space is given with
the property that the continuous spectrum of the global holomorphic functions
is not locally compact at any point.

Introduction. The spectrum $\mathcal{S}_c(\mathcal{O}(X))$ of the global holomorphic functions
$\mathcal{O}(X)$ on a complex space $(X, \mathcal{O})$ is the set of all continuous complex-valued alge-
bra homomorphisms on $\mathcal{O}(X)$ endowed with the Gelfand topology. This functional
analytic concept has important applications to fundamental problems in complex
analysis. For instance, due to a theorem of Igusa-Remmert-Iwahashi-Forster (see
[3, 1.5]), if there is at least one point where the spectrum $\mathcal{S}_c(\mathcal{O}(X))$ is not lo-
cally compact, then the complex space $(X, \mathcal{O})$ does not have a Stein envelope of
holomorphy.

In all the examples known of complex spaces $(X, \mathcal{O})$ without a Stein envelope of
holomorphy, the set of points in the spectrum $\mathcal{S}_c(\mathcal{O}(X))$ where local compactness
fails is a nonempty, extremely small, proper subset of $\mathcal{S}_c(\mathcal{O}(X))$; frequently, it
consists of just one point [2, 5.3; 7, 4.2, 4.3]. A natural question is whether such
pathological points can be described as a thin subset of the spectrum. However,
the purpose of this note is to show that the spectrum $\mathcal{S}_c(\mathcal{O}(X))$ of a complex space
$(X, \mathcal{O})$ need not be locally compact at any point at all; in the example constructed
here, $X$ is two dimensional. A complex space always refers to a reduced complex
space with countable topology.

Construction. The example will be constructed in two steps. The main idea
of the first step is to find a two-dimensional complex space $(X, \mathcal{O})$ whose spectrum
$\mathcal{S}_c(\mathcal{O}(X))$ has the following description. Take a two-dimensional complex plane
and attach infinitely many two-dimensional complex planes transversally along ev-
ery line of a countable dense subset of lines in the first plane. In the second step,
the unique position of the initial plane will be eliminated by carrying out the con-
struction of the first step for every plane attached to the initial plane.

First step of the construction. For every nonnegative integer $n \in \mathbb{N}$, set
$$S_n := \{(x, y) \in \mathbb{C}^2 : |x| \leq n + 1, n \leq |y| \leq n + \frac{1}{2}\}.$$ 

Let $D$ be the Reinhardt domain in $\mathbb{C}^2$ obtained by removing all the sets $S_n$, $n \in \mathbb{N},$
from $\mathbb{C}^2$. 

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Take countably many copies \( D_n, \, n \in \mathbb{N} \), of \( D \). Fix \( D_0 \), and for \( n \geq 1 \) consider each \( D_n \) as being transversal to \( D_0 \) in the following way. Choose a dense sequence \((\alpha_n)_{n \in \mathbb{N}}\) in \( \mathbb{C} \) with \( |\alpha_n| < n + 1 \) for \( n \in \mathbb{N} \). For \( \mathbb{N}^* := \mathbb{N} \setminus \{0\} \) let \( \varphi: \mathbb{N}^* \to \mathbb{N} \) be a surjective map where each value is assumed countably many times and \( \varphi(n) < n \) holds for every \( n \in \mathbb{N}^* \). The existence of such a map can be seen by taking for every \( n \in \mathbb{N}^* \) a \( p \in \mathbb{N} \) with \( p^2 \leq n < (p + 1)^2 \) and then defining \( \varphi(n) := n - p^2 \).

In \( \mathbb{C}^3 \) with coordinates \( x, \, y, \, z \), set \( D_0 := D \times \{0\} \). Let \( D_m \) be the Reinhardt domain in \( \{w(m)\} \times \mathbb{C}^2 \) which remains after removing for every \( n \in \mathbb{N} \) the set \( \{w(m)\} \times S_n', \) where

\[
S_n' := \{(y, z) \in \mathbb{C}^2 : n \leq |y| \leq n + \frac{1}{2}, \, |z| \leq n + 1\}.
\]

Roughly speaking, the desired complex space \( X \) is defined by attaching each \( D_n \) for \( n \geq 1 \) to \( D_0 \) along

\[
R_n := \{(x, y, 0) \in \mathbb{C}^3 : x = \alpha_{\varphi(n)}, \, n + \frac{1}{2} < |y| < n + 1\}.
\]

More precisely, observe that, for \( n \in \mathbb{N}^* \), \( R_n \) is always an analytic subset of \( D_n \); since \( |\alpha_{\varphi(n)}| < n + 1 \) is valid, \( R_n \) is also an analytic subset of \( D_0 \). For every \( n \in \mathbb{N}^* \), a complex space \( X_n \) having \( R_{n+1} \) as an analytic subset will be defined by induction.

Let \( X_1 \) be the complex space obtained by attaching \( D_1 \) to \( D_0 \) along \( R_1 \). This space

\[
X_1 := D_0 +_{R_1} D_1
\]

is the fiber sum (pushout) of \( D_0 \) and \( D_1 \) under the inclusion \( R_1 \to D_0 \) and \( R_1 \to D_1 \) [6]. Define

\[
X_n := X_{n-1} +_{R_n} D_n
\]

as the fiber sum of \( X_{n-1} \) and \( D_n \) under the inclusions \( R_n \to X_{n-1} \) and \( R_n \to D_n \).

For \( n \in \mathbb{N}^* \) let \( \iota_n: X_n \to X_{n+1} \) be the inclusion; \( \iota_n \) embeds \( X_n \) in \( X_{n+1} \) as a closed subspace. Denote by \( X \) the direct limit of the expanding system \((X_n, \iota_n)\), i.e.

\[
X := \lim_{\rightarrow} X_n.
\]

With the direct limit topology, \( X \) is a Hausdorff space [5, 4.1]. To see that \( X \) can be given a complex structure, note that there is an open covering \( \{U_n : n \in \mathbb{N}^*\} \) of \( X \) with the subsequent properties: For each \( n \in \mathbb{N}^* \), \( U_n \) is a subset of \( X_n \), and the complex charts on \( X \) defined by the \( U_n \) can be glued together to form a complex structure \( \mathcal{O} \) on \( X \) [4, VA7].

In order to calculate the spectrum of the algebra \( A := \mathcal{O}(X) \), take countably many copies \( \mathbb{C}^2, \, n \in \mathbb{N} \), of \( \mathbb{C}^2 \) such that the first copy \( \mathbb{C}^2 := \mathbb{C}^2 \times \{0\} \) is fixed and the other copies \( \mathbb{C}^2 := \{w(n)\} \times \mathbb{C}^2, \, n \geq 1 \), are transversal to the first copy. Let \( Y := \bigsqcup_{n \in \mathbb{N}} \mathbb{C}^2 \) denote their disjoint union with the natural complex structure. \( A \) is topologically isomorphic to

\[
\left\{(f_n)_{n \in \mathbb{N}} \in \prod_{n=0}^{\infty} \mathcal{O}(\mathbb{C}^2_n) : f_n \big|_{(z=0)} = f_0 \big|_{(z=w(n))}, \, n \geq 1 \right\}
\]

and can therefore be viewed as a function algebra on \( Y \).
LEMMA 1. Let \( \chi: Y \to S_c(A) \) be the canonical map assigning to every point \( y \in Y \) the point evaluation \( \hat{y} \) defined by \( \hat{y}(f) := f(y) \) for \( f \in A \). The spectrum \( S_c(A) \) is not locally compact at \( \hat{y} \) for any \( y \in C_0^2 \).

PROOF. Let \( Q \) denote the quotient of \( Y \) by the equivalence relation \( R_x \) associated to the map \( \chi \). \( Q \) is obtained by identifying the complex line \( \{ x = \alpha \varphi(n) \} \) in \( C_0^2 \) with the complex line \( \{ z = 0 \} \) in \( C_0^2 \) for every \( n \geq 1 \).

In \( C_0^2 \), along each line \( \{ x = \alpha \varphi(n) \} \) countably many planes are attached, because there are countably many preimages of \( \varphi(n) \). Consequently, \( C_0^2 \) is not locally compact along the line \( \{ x = \alpha \varphi(n) \} \) for every \( n \geq 1 \). Since the sequence \( (\alpha_n)_{n \in \mathbb{N}} \) is dense in \( C \), the plane \( C_0^2 \) is not locally compact along this line. Such lines are dense in \( C^2_{\varphi(n)} \), implying that \( C^2_{\varphi(n)} \) is not locally compact anywhere. If \( p: Y \to Q \) is the projection, then \( Q \) is not locally compact at \( p(y) \) for any \( y \in C_0^2 \).

Let \( \overline{\chi}: Q \to S_c(A) \) be the unique map making the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\chi} & S_c(A) \\
p \downarrow & & \downarrow \overline{\chi} \\
Q := Y/R_x
\end{array}
\]

commutative. Since \( \overline{\chi} \) is continuous, the assertion of Lemma 1 follows if \( \overline{\chi} \) is proper, i.e. inverse images of compact sets are compact.

To prove that the map \( \overline{\chi} \) really is proper, the fact that it is surjective will be used. This, in turn, can be seen as follows. \( A \) is the strongly dense inverse limit of the Fréchet algebras

\[
A_m := \left\{ (f_0, \ldots, f_m) \in \prod_{n=0}^{m} O(C^2_n): f_n \big|_{\{ x = 0 \}} = f_0 \big|_{\{ x = \alpha \varphi(n) \}}, \ 1 \leq n \leq m \right\}
\]

for \( m \in \mathbb{N} \) with respect to the surjective maps

\[
\pi_m: A_{m+1} \to A_m, \quad (f_0, \ldots, f_{m+1}) \mapsto (f_0, \ldots, f_m).
\]

According to a theorem of Arens [1, 5.21], the surjectivity of the projection \( \sigma_m: A \to A_m \) onto the first \( m + 1 \) components implies that, in the category of sets, \( S_c(A) \) is the direct limit of the system \( (S_c(A_m), \pi'_m) \) where \( \pi'_m: S_c(A_m) \to S_c(A_{m+1}) \) is the transposition \( \psi \mapsto \psi \circ \pi_m \). The spectrum \( S_c(A_m) \) is homeomorphic to the Stein space \( X'_m \) which is obtained from the disjoint union \( \bigsqcup_{n=0}^{m} C^2_n \) when the line \( \{ x = \alpha \varphi(n) \} \) in \( C_0^2 \) is identified with the line \( \{ z = 0 \} \) in \( C_0^2 \) for \( 1 \leq n \leq m \) [3, 1.5]. Thus, \( \overline{\chi} \) is surjective.

Now it can be shown that \( \overline{\chi} \) is proper. Let \( K \) be a compact subset of \( S_c(A) \). There is an \( m \in \mathbb{N} \) with

\[
(*) \quad K \subset O'_m(S_c(A_m)),
\]

where \( O'_m: S_c(A_m) \to S_c(A) \) is the transportation. If this were not the case, there would be a sequence \( (x_m)_{m \in \mathbb{N}} \) in \( K \) such that, when considered as a sequence in \( Q \) via the bijective map \( \chi \), the point \( x_m \) would lie in \( C_0^2 \) but not on the line \( \{ z = 0 \} \). Cartan's Theorem A would then ensure the existence of a function \( f_m \in O(C^2_m) \) with

\[
f_m(x_m) = m \quad \text{and} \quad f_m = 0 \quad \text{on} \ \{ z = 0 \}
\]
for every $m \geq 1$. Setting $f_0 := 0$, an element $f := (f_n)_{n \in \mathbb{N}}$ of $A$ would exist which is unbounded on the sequence $(x_m)_{m \in \mathbb{N}}$. This is a contradiction, since $f$ can be identified with its Gelfand transform which is bounded in $K$. Consequently, an $m \in \mathbb{N}$ exists with the property (§). Since $\sigma'_m$ is a topological embedding, $K$ can be considered as a compact subset of $X'_m$ and hence of $Q$, proving that $\check{\chi}$ is proper. This completes the proof of Lemma 1.

**Second step of the construction.** Let $(\alpha_n)_{n \in \mathbb{N}}$, $D$ and $D_0 := D \times \{0\}$ be as in the first step of the construction. Countably many copies $D_n$, $n \geq 1$, of $D$ which are either parallel or transverse to $D_0$ will be defined and attached together by means of a surjective map $\varphi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, $\varphi(0) = (0,0)$, $n \mapsto (\varphi_1(n), \varphi_2(n))$, where each value is assumed countably often and $\varphi_i(n) < n$ is true for every $n \in \mathbb{N}^*$ and for $i = 1, 2$. Such a map is given, for example, by composing the map $\mathbb{N}^* \to \mathbb{N}$ used in the first step with a bijective map $\psi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, $n \mapsto (\psi_1(n), \psi_2(n))$, satisfying $\psi_i(n) \leq n$ for $n \in \mathbb{N}$ and $i = 1, 2$.

For $n \geq 1$, $D_n$ will be defined by induction to be transverse to $D_{\varphi_1(n)}$. Let $D_1$ be as in the first step of the construction with $\alpha_0 := 0$. Call that part of $D_1$ in the $yz$-plane $D'$, i.e. $D_1 = \{0\} \times D'$. If $D_m$ has been defined to be parallel or transverse to $D_0$ and transverse to $D_{\varphi_1(m)}$ for $1 \leq m \leq n - 1$, then $D_{\varphi_1(n)}$ is either parallel or transverse to $D_0$, since $\varphi_1(n) \leq n - 1$. In the first case define $D_n := \{\alpha_2(n)\} \times D'$ and in the second case set $D_n := D \times \{\alpha_2(n)\}$.

The idea now is to obtain a complex space $X$ by identifying each $D_n$ with $D_{\varphi_1(n)}$ along

$$R_n := \{(x, y, z) \in \mathbb{C}^3 : x = \alpha_{\varphi_1(n)}, n + \frac{1}{2} < |y| < n + 1, z = \alpha_{\varphi_2(n)}\},$$

if $D_{\varphi_1(n)}$ is parallel to $D_0$ or along

$$R_n := \{(x, y, z) \in \mathbb{C}^3 : x = \alpha_{\varphi_2(\varphi_1(n))}, n + \frac{1}{2} < |y| < n + 1, z = \alpha_{\varphi_2(n)}\},$$

if $D_{\varphi_1(n)}$ is transverse to $D_0$. Because $|\alpha_{\varphi_2(\varphi_1(n))}| < n + 1$ and $|\alpha_{\varphi_2(n)}| < n + 1$ hold, $R_n$ is an analytic subset of $D_n$ as well as of $D_{\varphi_1(n)}$, and such an identification is possible [6].

To be more exact, $X$ will again denote the direct limit of expanding system of complex spaces $X_n$, $n \geq 1$, defined by induction as follows:

$$X_1 := D_0 + R_1, D_1, \quad X_n := X_{n-1} + R_n D_n.$$  

As before, $X$ is a complex space [5, 4].

The spectrum of $A := O(X)$ is determined by considering in $\mathbb{C}^3$ countably many copies $C_n^2$, $n \in \mathbb{N}$, of $\mathbb{C}^2$ parallel to $\mathbb{C}^2 \times \{0\}$ or to $\{0\} \times \mathbb{C}^2$ which are defined by induction. Let $C_0^2 := \mathbb{C}^2 \times \{0\}$ and $C_1^2 := \{0\} \times \mathbb{C}^2$. If $C_{\varphi_1(n)}^2$ has already been defined for $1 \leq m \leq n - 1$, then $C_{\varphi_1(n)}^2$ is either parallel or transverse to $C_0^2$. If the former is true, put $C_n^2 := \{\alpha_{\varphi_2(n)}\} \times \mathbb{C}^2$ and otherwise define $C_n^2 := C_1^2 \times \{0\} \times \{z = \alpha_{\varphi_2(n)}\}$. $A$ is topologically isomorphic to the set of elements $(f_n)_{n \in \mathbb{N}} \in \prod_{n=0}^{\infty} O(C_n^2)$ satisfying the following conditions for all $n \geq 1$:

$$f_n \bigg|_{\{x = \alpha_{\varphi_2(\varphi_1(n))}\}} = f_{\varphi_1(n)} \bigg|_{\{x = \alpha_{\varphi_2(n)}\}}$$

when $C_n^2$ is transverse to $C_0^2$ and otherwise

$$f_n \bigg|_{\{x = \alpha_{\varphi_2(\varphi_1(n))}\}} = f_{\varphi_1(n)} \bigg|_{\{z = \alpha_{\varphi_2(n)}\}}.$$
**Lemma 2.** The spectrum $S_c(A)$ is not locally compact anywhere.

**Proof.** Denote by $Y := \bigsqcup_{n \in \mathbb{N}} \mathbb{C}^2_n$ the disjoint union of the planes $\mathbb{C}^2_n$ with the natural complex structure; $A$ is a subalgebra of $\mathcal{O}(Y)$. Let $\psi: Y \to S_c(A)$ be the map $y \mapsto \hat{y}$ for $\hat{y}(f) := f(y)$. The quotient $Q$ of $Y$ by the equivalence relation $R_\chi$ given by $\chi$ is not locally compact at any point. To verify this, notice that $Q$ is obtained from $Y$ by identifying the line

$$\{z = \alpha \varphi_2(\varphi_1(n))\} \quad \text{resp.} \quad \{x = \alpha \varphi_2(\varphi_1(n))\}$$

in $\mathbb{C}^2_n$ with the line

$$\{x = \alpha \varphi_2(n)\} \quad \text{resp.} \quad \{z = \alpha \varphi_2(n)\}$$

in $\mathbb{C}^2_{\varphi_1(n)}$ for every $n \geq 1$; the choice of the line depends upon whether $\mathbb{C}^2_{\varphi_1(n)}$ is parallel or transverse to $\mathbb{C}^2_0$. Since there are countably many preimages of $\varphi(n)$, there are countably many planes attached to $\mathbb{C}^2_{\varphi_1(n)}$ along the line given by $x = \alpha \varphi_2(n)$ resp. $z = \alpha \varphi_2(n)$. Hence, $\mathbb{C}^2_{\varphi_1(n)}$ is not locally compact along this line. Such lines are dense in $\mathbb{C}^2_{\varphi_1(n)}$, implying the $\mathbb{C}^2_{\varphi_1(n)}$ is not locally compact anywhere. Because $\varphi$ is surjective, no plane $\mathbb{C}^2_n$ has a point at which it is locally compact. Thus, $Q$ is not locally compact anywhere.

Let $\bar{\chi}: Q \to S_c(A)$ be the canonical map induced by $\chi$. As in Lemma 1, it follows from a theorem of Arens [1, 5.21] that the continuous injective map $\bar{\chi}$ is surjective. Together with Cartan's Theorem A, this implies that $\bar{\chi}$ is proper, proving Lemma 2.

**References**