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OSCILLATION THEOREMS FOR SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH DAMPING

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ABSTRACT. In this paper, we present some criteria for the oscillation of the differential equation with damping

$$(\gamma(t)x'(t))' + p(t)x'(t) + q(t)x(t) = 0, \qquad t \in [t_0, \infty),$$

where $p(t)$ and $q(t)$ are allowed to change sign on $[t_0, \infty)$, and $\gamma(t) > 0$. One of our results is new even for the differential equations

$$x''(t) + q(t)x(t) = 0,$$

and

$$x''(t) + p(t)x'(t) + q(t)x(t) = 0.$$

In this paper, we study the oscillatory behavior of the solution of the second order differential equation with damping

$$(\gamma(t)x'(t))' + p(t)x'(t) + q(t)x(t) = 0, \tag{1}$$

where γ, p and q are continuous on $[t_0, \infty)$, $t_0 > 0$, $\gamma > 0$, and p and q are allowed to take on negative values for arbitrarily large t. The oscillatory character is considered in the usual sense, i.e., a solution of equation (1) is said to be oscillatory if it has arbitrarily large zeros; otherwise it is said to be nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.

In the absence of damping, there is a very large body of literature devoted to the corresponding equations

$$x''(t) + q(t)x(t) = 0, \tag{$*$}$$

$$(\gamma(t)x'(t))' + q(t)x(t) = 0. \tag{$**$}$$

Although (1) can be put in the forms ($*$) and ($**$) by multiplication by an integrating factor and, if necessary, by simple transformations, there are advantages in obtaining direct oscillation theorems for (1): besides the obvious practical advantage of eliminating the need for the integrating factor, there is an incentive in developing methods which will generalize to more general equations.

The use of averaging functions in the study of oscillation dates back to works of Wintner [**8**] and Hartman [**2**]. Coles [**1**] and Willett [**7**], and more recently, Kwong and Zettl [**4**] developed averaging techniques and, respectively, established more general theorems for equations ($*$) and ($**$) by considering weighted averages of the integral of q.

Recently, by exploiting more fully a simple "completing square" and averaging technique of Kamenev [**3**], the author [**9**] has given the following oscillation theorem for equation (1) with $\gamma(t) \equiv 1$.

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THEOREM. *If there exist $\alpha \in (1,\infty)$ and $\beta \in [0,1)$ such that*

$$\lim_{t\to\infty} \sup t^{-\alpha} \int_{t_0}^{t} (t-\tau)^{\alpha}\tau^{\beta} q(\tau)\, d\tau = \infty, \tag{2}$$

$$\lim_{t\to\infty} \sup t^{-\alpha} \int_{t_0}^{t} [(t-\tau)p(\tau)\tau + \alpha\tau - \beta(t-\tau)]^2 (t-\tau)^{\alpha-2}\tau^{\beta-2}\, d\tau < \infty, \tag{3}$$

then (1) *is oscillatory.*

Of particular interest, therefore, is the problem of finding criteria for the oscillation of (1) when (2) or (3) is not satisfied. In this paper we will establish two oscillation theorems. The first theorem considerably improves some known results and the second is new even for equations (*) and (**). Our results are as follows:

THEOREM 1. *Suppose that there exist a positive continuously differentiable function $h(t)$ on $[t_0,\infty)$ and a constant $\alpha \in (0,\infty)$ such that*

$$\lim_{t\to\infty} \sup t^{-\alpha} \int_{t_0}^{t} \left\{ (t-\tau)^{\alpha} h(\tau) q(t) - \frac{1}{4}\left[(t-\tau)\frac{h(t)p(t)}{\gamma(t)} + \alpha h(\tau) - (t-\tau)h'(\tau) \right]^2 \times (t-\tau)^{\alpha-2}\frac{\gamma(\tau)}{h(\tau)} \right\} d\tau = \infty, \tag{4}$$

then equation (1) *is oscillatory.*

PROOF. Let $x(t)$ be a nonoscillatory solution of (1). Without loss of generality, we suppose that $x(t) \neq 0$ for $t \geq t_0$. Furthermore, we put

$$\omega(t) = \gamma(t)x'(t)/x(t).$$

Then it follows from (1) that

$$\omega'(t) + \omega^2(t)/\gamma(t) + p(t)\omega(t)/\gamma(t) + q(t) = 0, \qquad t \geq t_0, \tag{5}$$

and consequently, for all $t > s \geq t_0$,

$$\int_s^t (t-\tau)^{\alpha} h(\tau)\omega'(\tau)\, dt + \int_s^t (t-\tau)^{\alpha} \frac{h(\tau)\omega^2(\tau)}{\gamma(\tau)}\, d\tau + \int_s^t (t-\tau)^{\alpha}\frac{h(\tau)p(\tau)\omega(\tau)}{\gamma(\tau)}\, d\tau + \int_s^t (t-\tau)^{\alpha} h(\tau) q(\tau)\, d\tau = 0.$$

Since

$$\int_s^t (t-\tau)^{\alpha} h(\tau)\omega'(\tau)\, d\tau = \alpha \int_s^t (t-\tau)^{\alpha-1} h(\tau)\omega(\tau)\, d\tau - \int_s^t (t-\tau)^{\alpha} h'(\tau)\omega(\tau)\, d\tau - \omega(s)(t-s)^{\alpha} h(s),$$

we obtain that

$$\begin{aligned}\int_s^t &(t-\tau)^\alpha h(\tau)q(\tau)\,d\tau\\ &= (t-s)^\alpha h(s)\omega(s) - \int_s^t \frac{(t-\tau)^\alpha h(\tau)\omega^2(\tau)}{\gamma(\tau)}\,d\tau\\ &\quad - \int_s^t \left[(t-\tau)\frac{h(\tau)p(\tau)}{\gamma(\tau)} + \alpha h(\tau) - (t-\tau)h'(\tau)\right](t-\tau)^{\alpha-1}\omega(\tau)\,d\tau,\end{aligned}$$

and hence

$$\begin{aligned}\int_s^t &\left\{(t-\tau)^\alpha h(\tau)q(\tau)\right.\\ &\qquad \left. - \frac{1}{4}\left[(t-\tau)\frac{h(\tau)p(\tau)}{\gamma(\tau)} + \alpha h(\tau) - (t-\tau)h'(\tau)\right]^2 (t-\tau)^{\alpha-2}\frac{\gamma(\tau)}{h(\tau)}\right\} d\tau\\ &= (t-s)^\alpha h(s)\omega(s) - \int_s^t \left\{(t-\tau)^{\alpha/2}\left(\frac{h(\tau)}{\gamma(\tau)}\right)^{1/2}\omega(\tau)\right.\\ &\qquad + \frac{1}{2}\left[(t-\tau)\frac{h(\tau)p(\tau)}{\gamma(\tau)} + \alpha h(\tau) - (t-\tau)h'(\tau)\right]\\ &\qquad \left.\times (t-\tau)^{(\alpha-2)/2}\left(\frac{\gamma(\tau)}{h(\tau)}\right)^{1/2}\right\}^2 d\tau\\ &\le (t-s)^\alpha h(s)\omega(s), \qquad s \ge t_0.\end{aligned}$$

Divide (7) by t^α and take the upper limit as $t \to \infty$. Using (4), we obtain a contradiction. This completes the proof of the theorem.

REMARK 1. Note that Theorem 1 does not even require $\int_{t_0}^\infty d\tau/\gamma(\tau) = \infty$ as does the Wintner-Leighton theorem [**8**, **6**], and does not require that the damping coefficient $p(t)$ is a "small" function.

REMARK 2. Theorem 1 above includes a result in [**9**] and Theorem 2 in [**10**] on equation (1) with $\gamma(t) = 1$.

The following corollary improves and generalizes Corollary 6 and Corollary 7 in [**5**].

COROLLARY 1. *If*

$$\limsup_{t\to\infty} t^{-2}\int_{t_0}^t \left[(t-\tau)^2 h(\tau)q(\tau) - \frac{(t-\tau)^2 h'^2(\tau)\gamma(\tau)}{4h(\tau)} - (t-\tau)h(\tau)\gamma'(\tau)\right] d\tau = \infty, \tag{8}$$

with $h(t)$ *as in Theorem* 1, *then equation* (**) *is oscillatory.*

PROOF. From (7), letting $p(t) \equiv 0$ and $\alpha = 2$, we have that

(9)

$$t^{-2}\int_{t_0}^t \left[(t-\tau)^2 h(\tau)q(\tau) - h(\tau)\gamma(\tau) - \frac{(t-\tau)^2 h'^2(\tau)\gamma(\tau)}{4h(\tau)} + (t-\tau)h'(\tau)\gamma(\tau)\right] d\tau \le t^{-2}(t-t_0)^2 h(t_0)\omega(t_0).$$

Since

$$\begin{aligned}\int_{t_0}^{t}(t-\tau)h'(\tau)\gamma(\tau)\,d\tau &= \int_{t_0}^{t}\int_{t_0}^{\tau} h'(\xi)\gamma(\xi)\,d\xi\,d\tau \\ &= \int_{t_0}^{t} h(\tau)\gamma(\tau)\,d\tau - \int_{t_0}^{t}(t-\tau)h(\tau)\gamma'(\tau)\,d\tau - h(t_0)\gamma(t_0)(t-t_0),\end{aligned}\tag{10}$$

(9) and (10) together contradict our hypothesis (8).

THEOREM 2. *Suppose that there exist a positive continuously differentiable function* $h(t)$ *on* $[t_0,\infty)$ *and* $\alpha\in(1,\infty)$ *such that*

$$\limsup_{t\to\infty} t^{-\alpha}\int_{t_0}^{t}(t-\tau)^{\alpha}h(\tau)q(\tau)\,d\tau<\infty,\tag{11}$$

and there exists a continuous function $\varphi(t)$ *on* $[t_0,\infty)$ *such that*

$$\begin{aligned}\liminf_{t\to\infty} t^{-\alpha}\int_{s}^{t}\Big\{&(t-\tau)^{\alpha}h(\tau)q(\tau) \\ &-\frac{1}{4}\left[(t-\tau)\frac{h(\tau)p(\tau)}{\gamma(\tau)}+\alpha h(\tau)-(t-\tau)h'(\tau)\right]^2 \\ &\times(t-\tau)^{\alpha_2}\frac{\gamma(\tau)}{h(\tau)}\Big\}\,d\tau\ge\varphi(s),\end{aligned}\tag{12}$$

and

$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{\varphi_+^2(\tau)}{h(\tau)\gamma(\tau)}\,d\tau=\infty,\tag{13}$$

where $\varphi_+(t)=\max(\varphi(t),0)$, *then equation* (1) *is oscillatory.*

PROOF. Suppose that $x(t)$ is a solution of (1) with $x(t)\neq 0$ for $t\ge t_0$. Set $\omega(t)=\gamma(t)x'(t)/x(t)$. As in the proof of Theorem 1, (7) holds. Dividing (7) by t^{α} and taking the lower limit as $t\to\infty$, we obtain $\varphi(s)\le h(s)\omega(s)$, $s\ge t_0$, which implies that

$$\varphi_+^2(s)\le h^2(s)\omega^2(s).\tag{14}$$

We define functions

$$u(t)=t^{-\alpha}\int_{t_0}^{t}\left[(t-\tau)\frac{h(\tau)p(\tau)}{\gamma(\tau)}+\alpha h(\tau)-(t-\tau)h'(\tau)\right](t-\tau)^{\alpha-1}\omega(\tau)\,d\tau,\qquad t>t_0,$$

$$v(t)=t^{-\alpha}\int_{t_0}^{t}(t-\tau)^{\alpha}h(\tau)\frac{\omega^2(\tau)}{\gamma(\tau)}\,d\tau,\qquad t>t_0.$$

From (6),

$$u(t)+v(t)=t^{-\alpha}(t-t_0)^{\alpha}h(t_0)\omega(t_0)-t^{-\alpha}\int_{t_0}^{t}(t-\tau)^{\alpha}h(\tau)q(\tau)\,d\tau,\tag{15}$$

and we observe that (12) implies that

$$\liminf_{t\to\infty} t^{-\alpha}\int_{s}^{t}(t-\tau)^{\alpha}h(\tau)q(\tau)\,d\tau\ge\varphi(s),\qquad s\ge t_0,\tag{16}$$

and

$$\limsup_{t\to\infty} t^{-\alpha}\int_{t_0}^{t}(t-\tau)^{\alpha}h(\tau)q(\tau)\,d\tau - \liminf_{t\to\infty}\frac{t^{-\alpha}}{4}\int_{t_0}^{t}\left[(t-\tau)\frac{h(\tau)p(\tau)}{\gamma(\tau)}+\alpha h(\tau)-(t-\tau)h(\tau)\right]^2 \times (t-\tau)^{\alpha-2}\frac{\gamma(\tau)}{h(\tau)}\,d\tau \geq \varphi(t_0). \tag{17}$$

(17) together with (11) shows that there exists a sequence

$$\{t_n\}_1^{\infty},\quad t_n > t_0,\quad n=1,2,3,\dots,\quad \lim_{n\to\infty} t_n=\infty, \tag{18}$$

such that

$$\lim_{n\to\infty}\frac{t_n^{-\alpha}}{4}\int_{t_0}^{t_n}\left[(t_n-\tau)\frac{h(\tau)p(\tau)}{\gamma(\tau)} + \alpha h(\tau)-(t_n-\tau)h'(\tau)\right]^2(t_n-\tau)^{\alpha-2}\frac{\gamma(\tau)}{h(\tau)}\,d\tau<\infty. \tag{19}$$

Taking the upper limit as $t\to\infty$ in (15) and using (16), we have

$$\begin{aligned}\limsup_{t\to\infty}\{u(t)+v(t)\} &= h(t_0)\omega(t_0) - \liminf_{t\to\infty} t^{-\alpha}\int_{t_0}^{t}(t-\tau)h(\tau)q(\tau)\,d\tau \\ &= h(t_0)\omega(t_0)-\varphi(t_0)=k.\end{aligned} \tag{20}$$

Hence for all sufficiently large n,

$$u(t_n)+v(t_n)<k. \tag{21}$$

Since

$$v(t)=\int_{t_0}^{t}\left(1-\frac{\tau}{t}\right)^{\alpha}h(\tau)\frac{\omega^2(\tau)}{\gamma(\tau)}\,d\tau>0$$

is increasing in $t>t_0$, we see that $\lim_{t\to\infty}v(t)=c$, where $c=\infty$ or is a positive constant. Suppose that $c=\infty$, then $\lim_{n\to\infty}v(t_n)=\infty$ and, by (21),

$$\lim_{n\to\infty}u(t_n)=-\infty. \tag{22}$$

(21) and (22) lead to $u(t_n)/v(t_n)+1<\varepsilon$, where $0<\varepsilon<1$ is a constant, that is,

$$u(t_n)/v(t_n)<\varepsilon-1<0,\quad \text{for all large } t_n. \tag{23}$$

One the other hand, by the Schwarz inequality we have

$$\begin{aligned}0 &\leq t_n^{-2\alpha}\left(\int_{t_0}^{t_n}\left[(t_n-\tau)\frac{h(\tau)p(\tau)}{\gamma(\tau)}+\alpha h(\tau)-(t_n-\tau)h'(\tau)\right]^{\alpha-1}\omega(\tau)\,d\tau\right)^2 \\ &\leq \left(t_n^{-\alpha}\int_{t_0}^{t_n}\left[(t_n-\tau)\frac{h(\tau)p(\tau)}{\gamma(\tau)}+\alpha h(\tau)-(t_n-\tau)h'(\tau)\right]^2(t_n-\tau)^{\alpha-2}\frac{\gamma(\tau)}{h(\tau)}\,d\tau\right) \\ &\quad\cdot\left(t_n^{-\alpha}\int_{t_0}^{t_n}(t_n-\tau)^{\alpha}h(\tau)\frac{\omega^2(\tau)}{\gamma(\tau)}\,d\tau\right),\end{aligned}$$

for all large t_n, and so

$$0 \le \frac{u^2(t_n)}{v(t_n)}$$

$$\le t_n^{-\alpha} \int_{t_0}^{t_n} \left[(t_n - \tau) \frac{h(\tau)p(\tau)}{\gamma(\tau)} + \alpha h(\tau) - (t_n - \tau)h'(\tau) \right]^2 (t_n - \tau)^{\alpha-2} \frac{\gamma(\tau)}{h(\tau)} \, d\tau.$$

By (19), we have

$$0 \le \lim_{n\to\infty} \frac{u^2(t_n)}{v(t_n)}$$

$$\le \lim_{n\to\infty} t_n^{-\alpha} \int_{t_0}^{t_n} \left[(t_n - \tau) \frac{h(\tau)p(\tau)}{\gamma(\tau)} + \alpha h(\tau) - (t_n - \tau)h'(\tau) \right] (t_n - \tau)^{\alpha-2} \frac{\gamma(\tau)}{h(\tau)} \, d\tau$$

$$< \infty.$$

which contradicts (22) and (23).

Hence $\lim_{t\to\infty} v(t) = c < \infty$. Using (14), we then obtain that

$$\lim_{t\to\infty} t^{-\alpha} \int_{t_0}^{t} (t-\tau)^{\alpha} \frac{\varphi_+^2(\tau)}{h(\tau)\gamma(\tau)} \, d\tau$$

$$\le \lim_{t\to\infty} t^{-\alpha} \int_{t_0}^{t} (t-\tau)^{\alpha} h(\tau) \frac{\omega^2(\tau)}{\gamma(\tau)} \, d\tau = c < \infty,$$

which contradicts condition (13). This completes the proof of Theorem 2.

REMARK 3. In the conditions of Theorem 2, $q(t)$ is not required to be integrable or bounded on $[t_0, \infty)$. See Examples 2 and 3 below.

EXAMPLE 1. Consider the equation

$$\left(\frac{1}{t}x'(t)\right)' + \sin t x'(t) + t^2 \cos t x(t) = 0, \qquad t \ge t_0 > 0. \tag{24}$$

If we take $h(t) = t$ and $\alpha = 2$, then all the hypotheses of Theorem 1 are satisfied. Hence (24) is oscillatory, while oscillation criteria in [**3**, **9** and **10**] fail to apply to equation (24).

EXAMPLE 2. Consider the equation

$$(t^{\lambda} x'(t))' + t^{\mu} \sin t x'(t) + t^{\nu} \cos t x(t) = 0, \qquad t \ge t_0 > 0, \tag{25}$$

where $-1 \le \lambda < 1$, $-\infty < \mu \le -1$ and $-1 < \nu \le 1$ are constants and $2\nu + 1 \ge \lambda$. Taking $h(t) = 1$ and $\alpha = 2$, we have

$$\lim_{t\to\infty} \sup t^{-2} \int_{t_0}^{t} (t-\tau)^2 \tau^{\nu} \cos\tau \, d\tau = -t_0^{\nu} \sin t_0 < \infty,$$

$$\lim_{t\to\infty} \inf t^{-2} \int_{s}^{t} \left\{ (t-\tau)^2 \tau^{\nu} \cos\tau \, d\tau - \frac{1}{4}[(t-\tau)\tau^{\mu} \sin\tau + 2]^2 \tau^{\lambda} \right\} d\tau$$

$$\ge -s^{\nu} \sin s - k, \qquad s \ge t_0,$$

where k is a positive constant. Set $\varphi(s) = -s^{\nu}\sin s - k$, there is an integer N such that $(2N+1)\pi + \pi/4 > t_0$, and when $n \geq N$ and $(2n+1)\pi + \pi/4 \leq s \leq 2(n+1)\pi - \pi/4$, $\varphi(s) = -s^{\nu}\sin s - k \geq \varepsilon s^{\nu}$, where ε is a small constant. Noting $2\nu - \lambda \geq -1$, we have

$$\begin{aligned}\lim_{t\to\infty}\int_{t_0}^{t}\frac{\varphi_+^2(s)}{s^{\lambda}}\,ds &\geq \sum_{N=n}^{\infty}\varepsilon^2\int_{(2n+1)\pi+\pi/4}^{2(n+1)\pi-\pi/4} s^{2\nu-\lambda}\,ds \\ &\geq \sum_{N=n}^{\infty}\varepsilon^2\int_{(2n+1)\pi+\pi/4}^{2(n+1)\pi-\pi/4}\frac{ds}{s} = \infty.\end{aligned}$$

Hence (25) is oscillatory by Theorem 2, whereas known of the none criteria can cover this result.

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