

**SOME EXAMPLES OF INDEFINITE COMPLETE COMPLEX  
 EINSTEIN HYPERSURFACES NOT LOCALLY SYMMETRIC**

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**ABSTRACT.** Some examples of indefinite complete complex Einstein hypersurfaces of an indefinite complex flat space which are not locally symmetric are given.

**Introduction.** Recently [3], a systematic study of complex Einstein hypersurfaces in indefinite complex space forms (see [1]) has been made as an extension of the known work of Smyth [5] to indefinite Kaehler metrics. In that paper, the authors found that, except in one case, indefinite complex Einstein hypersurfaces are locally symmetric.

The purpose of this paper is to give some examples of indefinite complex Einstein hypersurfaces of the indefinite complex flat space  $C_n^{2n+1}$ ,  $n > 1$  (see [1]), which are not locally symmetric. In fact, let  $M_p$ ,  $p \in \mathbf{Z}$ ,  $p \geq 2$ , be the complex hypersurface of  $C_n^{2n+1}$  given by

$$\sum_{j=1}^n (z_j - z_{n+j})^p = pz_{2n+1}.$$

Thus, we have a family of spaces  $M_p$  with several interesting features that we prove in the following

**THEOREM.** (a) Every  $M_p$  is a complete complex hypersurface of  $C_n^{2n+1}$  with index  $2n$ , which is Ricci-flat but nonflat.

(b)  $M_p$ , with  $p > 2$ , is not locally symmetric and  $M_2$  is.

(c) If  $p \neq p'$  then  $M_p$  and  $M_{p'}$  are not isometric, but each  $M_p$  is holomorphically diffeomorphic to  $C^{2n}$ .

(d) If  $p > 2$ , then  $M_p$  is not homogeneous with respect to the induced Kaehler metric.

We shall use here the same notation as in [3].

**1. Preliminaries.** For each  $z = (z_1, \dots, z_{2n+1}) \in M_p$

$$\xi_z = ((\bar{z}_1 - \bar{z}_{n+1})^{p-1}, \dots, (\bar{z}_n - \bar{z}_{2n})^{p-1}, (\bar{z}_1 - \bar{z}_{n+1})^{p-1}, \dots, (\bar{z}_n - \bar{z}_{2n})^{p-1}, 1)$$

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is a normal vector to  $M_p$  in  $z$  and satisfies  $g(\xi_z, \xi_z) = 1$ , where  $g$  is the usual Kaehler flat metric of index  $2n$  on  $\mathbb{C}_n^{2n+1}$  (see [1]). Thus,  $\xi$  is a unit normal vector field to  $M_p$ .

It is easy to see that the Weingarten endomorphism  $A$  associated to  $\xi$  is given, in each  $z \in M_p$ , by

$$(1.1) \quad \begin{aligned} A(a_1, \dots, a_n, a_{n+1}, \dots, a_{2n}, a_{2n+1}) \\ = (f_1(z)(\bar{a}_1 - \bar{a}_{n+1}), \dots, f_n(z)(\bar{a}_n - \bar{a}_{2n}), \\ f_1(z)(\bar{a}_1 - \bar{a}_{n+1}), \dots, f_n(z)(\bar{a}_n - \bar{a}_{2n}), 0) \end{aligned}$$

where  $f_j(z) = (1 - p)(\bar{z}_j - \bar{z}_{n+j})^{p-2}$ ,  $j = 1, 2, \dots, n$ .

Let  $P_j$  be the tensor field of type  $(1, 1)$  defined by

$$(1.2) \quad \begin{aligned} P_j(a_1, \dots, a_n, a_{n+1}, \dots, a_{2n}, a_{2n+1}) \\ = (0, \dots, 0, \bar{a}_j - \bar{a}_{n+j}, 0, \dots, 0, \bar{a}_j - \bar{a}_{n+j}, 0, \dots, 0) \end{aligned}$$

for  $j = 1, 2, \dots, n$ , where  $(a_i)$  denotes any tangent vector to  $M_p$  in  $z$ . Then, we can write

$$A = \sum_{j=1}^n f_j(z) P_j.$$

Moreover, the following properties hold:

$$(1.4) \quad \begin{aligned} P_j \text{ is a selfadjoint operator of the tangent space to } M_p \text{ in } z. \\ P_i \cdot P_j = 0, \quad \nabla P_j = 0 \quad \text{and} \quad J \cdot P_j = -P_j \cdot J \end{aligned}$$

where  $\nabla$  and  $J$  are, respectively, the metric connection and the complex structure of  $M_p$ .

By using the Gauss equation [3, (1.1)] for the curvature tensor  $R$  of  $M_p$ , we obtain

$$(1.5) \quad R = \sum_{i,j=1}^n f_{ij}(z) R_{ij}^0 + \sum_{i,j=1}^n h_{ij}(z) J R_{ij}^0$$

where  $f_{ij}(z)$  and  $h_{ij}(z)$  are, respectively, real and imaginary parts of  $\overline{f_i(z)} f_j(z)$ ,  $\overline{f_i(z)}$  is the complex conjugate of  $f_i(z)$  and  $R_{ij}^0$  the tensor field given by

$$(1.6) \quad \begin{aligned} R_{ij}^0(X, Y)Z = g(P_i Y, Z) P_j X - g(P_i X, Z) P_j Y \\ + g(J P_i Y, Z) J P_j X - g(J P_i X, Z) J P_j Y \end{aligned}$$

for all  $X, Y, Z$  tangent to  $M_p$ .

Taking into account that  $\nabla P_j = 0$ , it follows  $\nabla R_{ij}^0 = 0$  for all  $i, j = 1, 2, \dots, n$ . So,

$$(1.7) \quad \nabla_V^k R = \sum_{i,j=1}^n V \left( \begin{matrix} \dots \\ \dots \\ V(f_{ij}) \end{matrix} \right) R_{ij}^0 + \sum_{i,j=1}^n V \left( \begin{matrix} \dots \\ \dots \\ V(h_{ij}) \end{matrix} \right) J R_{ij}^0$$

for all  $V$  tangent to  $M_p$ ,  $k \in \mathbb{Z}$ ,  $k \geq 1$ , where  $\nabla_V^k = \nabla_V \dots \nabla_V^{(k)}$ .

**2. Proof of the Theorem.** From (1.1) we have  $A^2 = 0$  but  $M_p$  is not totally geodesic, and hence, the formula for the Ricci tensor (1.4) in [3] implies that  $M_p$  is Ricci flat but nonflat. Now suppose that  $\gamma = \gamma(t)$  is a geodesic in  $M_p$ . Then components of  $\gamma$  (as a curve in  $C_n^{2n+1}$ ) satisfy  $\gamma_j''(t) = \gamma_{n+j}''(t) = F_j(t)$ ,  $j = 1, 2, \dots, n$ ,  $\gamma_{2n+1}''(t) = G(t)$ , where  $F_j(t)$ ,  $G(t)$  are certain polynomial functions of complex coefficients with degree  $(F_j(t)) = 2p - 3$  and degree  $(G(t)) = p - 2$ . Thus, components of  $\gamma(t)$  are also polynomial and hence  $\gamma(t)$  is defined for all  $t \in R$ . This proves that  $M_p$  is complete for all  $p$ , and so we have (a).

For (b) note that functions  $f_{ij}$  and  $h_{ij}$  in (1.5) are polynomial of degree  $2(p - 2)$  (in particular constants for  $p = 2$ ). By using (1.7), we have  $\nabla^{2(p-2)+1}R = 0$ . So,  $\nabla R = 0$  for  $p = 2$ , that is,  $M_2$  is locally symmetric. Now, let us suppose  $p > 2$  and take for fixed  $j$ ,  $X \in \text{Ker } P_i$ ,  $i \neq j$ ,  $X \notin \text{Ker } P_j$ . Then, using (1.4), (1.7) becomes

$$(2.1) \quad g((\nabla_V^{2(p-2)}R)(X, JX)JX, X) \\ = -2\left(V\left(\begin{smallmatrix} 2(p-2) \\ \dots \end{smallmatrix} V(f_{jj})\right)\right)\{g(P_jX, X)^2 + g(JP_jX, X)^2\}.$$

If  $\nabla^{2(p-2)}R = 0$ , from (2.1) we obtain  $g(P_jX, X) = g(JP_jX, X) = 0$  because

$$V\left(\begin{smallmatrix} 2(p-2) \\ \dots \end{smallmatrix} V(f_{jj})\right) \neq 0.$$

But, from (1.2) this implies that  $X \in \text{Ker } P_j$  and it contradicts our assumption.

From the above discussion, we have  $\nabla^{2(p-2)+1}R = 0$  and  $\nabla^{2(p-2)}R \neq 0$  for all  $p$ . Then  $M_p$  and  $M_{p'}$  cannot be isometric if  $p \neq p'$ . However,  $M_p$  is the graph of a holomorphic function of  $C^{2n}$  on  $C$ . And this concludes (c).

Finally, we will prove (d). In fact, let us take  $z \in M_p$ ,  $p > 2$ , verifying  $z_j = z_{n+j}$  for all  $j = 1, 2, \dots, n$  and  $z_{2n+1} = 0$ . From (1.5) we have  $R = 0$  in this point  $z$ .  $M_p$  homogeneous implies  $R = 0$  in every point, but it is not possible because  $M_p$  is not a flat space.

**REMARKS.** (1) We know that  $M_p$  has index  $2n$  and real dimension  $4n$ . The change of the Kaehler metric  $g$  of the ambient space by its negative permits us to obtain that  $M_p$  can be also considered as a complex hypersurface of  $C_{n+1}^{2n+1}$  and so, we can give a similar result to our Theorem for the family  $M_p$  in  $C_{n+1}^{2n+1}$ .

(2)  $M_2$  is identical to the space of Example 3.4 in [3] which has a Weingarten endomorphism with maximal rank in each point (see also [2, p. 346]).

(3) Given positive integers  $s, m, s \leq m$ , we have that  $M_p \times C_s^m$  is a complex hypersurface of  $C_{n+s}^{2n+m+1}$  (or  $C_{n+s+1}^{2n+m+1}$ ) for all  $p \in Z, p \geq 2$ . Thus, we can obtain a result analogous to our Theorem for this family of spaces.

(4) In the definite case, Smyth proves in [6] that a complex hypersurface of a complex space form is homogeneous if and only if it is an Einstein space. This is not true here because our Theorem provides us many indefinite complete complex Einstein hypersurfaces that are not homogeneous.

(5) Since every  $M_p$  satisfies  $A^2 = 0$ , it is easy to see that the curvature tensor  $R$  of  $M_p$  verifies Nomizu's condition  $R \cdot R = 0$  in [4]. However, we know by (b) that if  $p > 2$ ,  $M_p$  is not locally symmetric. So,  $R \cdot R = 0$  does not imply  $\nabla R = 0$  for complete complex hypersurfaces of an indefinite complex flat space.

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