PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 98, Number 2, October 1986

EXAMPLES OF TOPOLOGICAL GROUPS HOMEOMORPHIC TO l_2^f

TADEUSZ DOBROWOLSKI

ABSTRACT. We prove that the following spaces are homeomorphic to l_2^f: (1) the group of piecewise continuous autotransformations of $[-1,1]$ preserving Lebesgue measure, and (2) certain subgroups obtained as group spans of linearly independent arcs in linear spaces. These are consequences of our discussion of the problem whether σ-fd-compact locally contractible metric groups must be either finite-dimensional or locally homeomorphic to l_2^f.

1. Introduction. We are concerned with the question of whether every infinite-dimensional, σ-fd-compact (the countable union of finite-dimensional compacta), contractible (resp. locally contractible) metrizable topological group G is homeomorphic (resp. locally homeomorphic) to l_2^f. Here l_2^f stands for the linear subspace consisting of all finite sequences of the Hilbert space l_2. According to a result of Mogilski **[15]**, a σ-fd-compact metric space is (locally) homeomorphic to l_2^f if and only if the following three conditions are met:

(1) X is an AR (an ANR).

(2) X satisfies the discrete approximation property.

(3) X is strongly universal for finite-dimensional compacta.

To use this result, we first note

PROPOSITION 1. *Every (locally) contractible, σ-fd-compact, metrizable topological group is an AR (an ANR).*

Proposition 1 is a consequence of the observation that contractible groups are also locally contractible [7] and a result of Haver **[11]** that σ-fd-compact locally contractible spaces are ANRs.

A space X has *the discrete approximation property* if each map $f\colon \oplus_1^\infty I^n \to X$ of the discrete union of n-dimensional cubes can be strongly approximated by $g\colon \oplus_1^\infty I^n \to X$ such that the collection $\{g(I^n)\}_{n=1}^\infty$ is discrete; see **[17]**. Using a result of K. H. Hofmann **[12]** concerning a local topological structure of locally compact groups and a result of **[8]** we are able to prove

PROPOSITION 2. *Let G be an infinite-dimensional metrizable topological group. If G is strongly LC^∞ (i.e. every neighborhood U of 1 contains a neighborhood $V \subset U$ of 1 such that spheres of all dimensions in V are contractible in U), then G satisfies the discrete approximation property.*

Received by the editors May 30, 1985.

1980 *Mathematics Subject Classification*. Primary 22A05; Secondary 57N20.

Key words and phrases. σ-finite-dimensional metric groups, contractible groups, equiconnecting maps, measure preserving transformations, group spans, linearly independent arcs in linear spaces.

A space X is *strongly universal for finite-dimensional compacta* [**15**] if every map of a finite-dimensional compactum into X which restricts to an embedding on some subcompactum can be approximated by an embedding into X extending the restricted embedding. By Propositions 1 and 2 the question we are concerned with reduces to detecting the strong universal property of G. We do not provide an answer in general. We are not even able to prove or disprove whether G is universal for all finite-dimensional (or two-dimensional) compacta. In some instances, however, dealing with "concrete" groups it is possible to verify the strong universal property. In §§4 and 5 we show that (1) the group of piecewise continuous bijections that preserve Lebesgue measure on the interval $[-1, 1]$, and (2) certain additive subgroups obtained as group spans of linearly independent arcs in metric linear spaces are homeomorphic to l_2^f. The proofs rely on a skeletal version of the strong universal property adopted from the case of convex sets [**6**] to the case of contractible groups. The natural equiconnected structure in convex sets is replaced by an equiconnected structure in contractible groups induced by a contracting homotopy.

2. Proof of Proposition 2. The proof is an immediate consequence of the two facts below; the proof of the second one is contained in [**8**].

LEMMA 1. *Let G be a metrizable topological group. If G is strongly LC^∞ and contains a neighborhood totally bounded in the right structure of G, then G is finite-dimensional.*

LEMMA 2. *Let G be a locally path-connected metrizable topological group. If G contains no totally bounded neighborhoods of* 1 *in the right structure of G, then G satisfies the discrete approximation property.*

PROOF OF LEMMA 1. Let d be a right-invariant metric on G. Write $d^*(x, y) = d(x, y) + d(x^{-1}, y^{-1})$, $x, y \in G$. It is known that the completion $\tilde{G}$ of the group G with respect to the metric d^* is a complete-metrizable topological group (see [**14**, p. 212]). Now assume U_0 is a symmetric neighborhood of 1 in G that is totally bounded with respect to d and consequently its closure U in $\tilde{G}$ is symmetric and totally bounded in the right structure of $\tilde{G}$. We infer from Lemma 2 of [**8**] that $\tilde{G}$ is locally compact. We shall show below that $\tilde{G}$ is strongly LC^∞. If we assume this, then by a result of K. H. Hofmann [**12**] stating that strongly LC^∞ locally compact groups are finite-dimensional $\tilde{G}$, and consequently G, is finite-dimensional.

Claim (cf. [**9**]). If $X_0 \subset X$ is a dense set which is strongly LC^∞ uniformly with respect to a metric of X, then X is also strongly LC^∞.

PROOF. Since X_0 is strongly LC^∞ uniformly with respect to a metric of X, for every $x \in X$ and each neighborhood U in X, there exists an open neighborhood $V \subset U$ of x such that spheres of all dimensions in $X_0 \cap V$ contract in $X_0 \cap U$. Let $\phi\colon \partial I^k \to V$ be a map of k-dimensional sphere. By a result of Toruńczyk [**16**], $X \setminus X_0$ is locally homotopy negligible in X. In particular, there exists a homotopy $(\phi_t)\colon \partial I^k \to V$ such that $\phi_0 = \phi$ and $\phi_1(\partial I^k) \subset X_0 \cap V$. Hence ϕ_1 can be homotopied to a point in $X_0 \cap U$.

COROLLARY 1. *Let G be a metric separable topological group. If G is an ANR, then G can be embedded as a dense subspace of a Hilbert space manifold M such that $M \setminus G$ is locally homotopy negligible in M.*

PROOF. First, note that the following converse of the assertion of Lemma 1 holds. If an LC^∞ metrizable topological group G is finite-dimensional, then it contains a neighborhood of 1 totally bounded in the right structure of G. If this was not true, then by Lemma 2 G would satisfy the discrete approximation property. The latter would imply the existence of ε-maps of n-dimensional cubes into G for arbitrary n and $\varepsilon > 0$ (see [**5**, Proposition 6.5]). Therefore G would be infinite-dimensional.

Hence, if G is finite-dimensional, then by the proof of Lemma 1, the completion $\tilde{G}$ is a strongly LC^∞, locally compact topological group. Moreover, $\tilde{G} \setminus G$ is locally homotopy negligible. The result of Hofmann implies that $\tilde{G}$ is finite-dimensional. Consequently, $\tilde{G}$ is a Lie group. On the other hand, if G is infinite-dimensional and d is a right invariant metric on G, then by Lemma 1 no neighborhood of 1 is totally bounded in the metric d. Let Y be the completion of (G, d). By a result of Toruńczyk [**16**], there exists an ANR $Z \subset Y$, so that Z is a G_δ-set in Y and $Z \setminus G$ is locally homotopy negligible. By Lemma 1 of [**8**], Z satisfies the discrete approximation property. Applying Toruńczyk's characterization of l_2-manifolds [**17**], we infer that Z is homeomorphic to a Hilbert space manifold.

Corollary 1 corresponds to a result of Curtis [**4**], proved in its final version of Bowers [**2**], stating that nowhere totally compact separable metric spaces densely embed into l_2. It would be useful to characterize these spaces (necessary nowhere locally compact ANRs) that are remainders of locally homotopy negligible sets in Hilbert space (or Hilbert cube) manifolds. Observe that a complete metrizable AR X is a remainder of a locally homotopy negligible set in a Hilbert cube (Hilbert space) iff X is homeomorphic to an infinite-dimensional convex set (X is homeomorphic to l_2). Let us finally note that we do not know whether there exists a space M of Corollary 1 that carries a topological group structure.

3. Skeletoids in contractible groups. Following [**6**], we say that a tower of subsets $X_1 \subset X_2 \subset \cdots$ in X is *strongly universal for finite-dimensional compacta* if, for every pair of finite-dimensional compacta (A, B) and every map $f\colon A \to X$ such that $f|B$ is an embedding into some X_m and every $\varepsilon > 0$, there exists an embedding $v\colon A \to X_n$ for some $n \geqslant m$ such that $v|B = f|B$ and $d(f, v) \leqslant \varepsilon$. We refer to $\bigcup_1^\infty X_i$ as a *skeletoid* in X. We will deal with skeletoids in spaces admitting an equiconnected structure. Let us recall that a homotopy $(c_t)\colon V \to X$ defined on a neighborhood $V = \operatorname{dom}(c_t)$ ($\operatorname{dom}(c_t)$ = domain of the homotopy (c_t)) of the diagonal in $X \times X$ is called a *local equiconnecting map* if $c_0(x_1, x_2) = x_1$, $c_1(x_1, x_2) = x_2$, and $c_t(x, x) = x$ for all $(x_1, x_2) \in V$, $x \in X$ and $t \in [0,1]$. The word "local" is dropped if $V = X \times X$.

In Proposition 3 we extend the method of constructing skeletoids in convex sets [**6**] to spaces admitting a local equiconnecting map. Following Curtis [**5**], a tower (X_i) of X is *finitely expansive* if for each m there exists a one-to-one map $u_m\colon X_m \times [0,1] \to X_n$ for some $n \geqslant m$ satisfying $u_m(x, 0) = x$ for all $x \in X_m$.

Proposition 3. *Let X be a metric space and let (c_t) be a local equiconnecting map on X. Suppose that* (i) *(X_i) is a finitely expansive tower in X,* (ii) *$\bigcup_1^\infty X_i$ is dense in X, and* (iii) *for each m the condition $x_1, x_2 \in \operatorname{dom}(c_t)$, $(x_1, x_2) \in X_m \times X_m$ implies $c_t(x_1, x_2) \in X_n$ for some $n \geqslant m$ and all t. Then (X_i) is strongly universal for finite-dimensional compacta.*

We will employ two lemmas; the proofs will be given following that of Proposition 3.

Lemma 3. *For every map $f\colon A \to X$ of a finite-dimensional compactum and every $\varepsilon > 0$, there exists for some n a map $\bar{f}\colon A \to X_n$ with $d(f, \bar{f}) < \varepsilon$.*

Lemma 4. *Every map $f\colon B \to X_m$ of a closed subset B of a finite-dimensional compactum A extends to a map $\bar{f}\colon U \to X_n$ for some $n \geqslant m$, where U is a neighborhood of B in A.*

Proof of Proposition 3. Given a map $f\colon A \to X$ of a finite-dimensional compactum A and a closed subset B of A such that $f|B\colon B \to X_m$ is an embedding, by Lemma 3, f can be approximated by $f_1\colon A \to X_{n_1}$ for some $n_1 \geqslant m$. Lemma 4 yields an extension $f_2\colon U \to X_{n_2}$, $n_2 \geqslant n_1$, of $f|B$. Pick a map $\lambda\colon A \to [0,1]$ with $\lambda|B = 1$ and $\lambda = 0$ outside a suitable small neighborhood of B in A. Then, clearly the map $\bar{f}(a) = c_{\lambda(a)}(f_1(a), f_2(a))$, $a \in A$, closely approximates f, and satisfies $\bar{f}|B = f|B$ and $\bar{f}(A) \subset X_k$ for some $k \geqslant m$. Let $e\colon A \to I^p$ be a map such that $e^{-1}\{(0,0,\ldots,0)\} = B$ and $e|A \setminus B$ is one-to-one. Since (X_i) is finitely expansive, there exists

$$u\colon X_k \times [0,1]^p \to X_n, \qquad n \geqslant k,$$

so that for all $x \in \bar{f}(A)$, $\operatorname{diam}(u(\{x\} \times I^p))$ is small and $u(x,(0,0,\ldots,0)) = x$. Set $v(a) = u(\bar{f}(a), e(a))$, $a \in A$. Clearly v is a one-to-one map into X_n approximating $\bar{f}$ and extending $\bar{f}|B = f|B$.

Proofs of Lemmas 3 and 4 are routine. The second one requires repeating a Dugundji argument (cf. [**3**, Theorem 2.4]) and will be omitted. We include the first one to emphasize the assumption that $\bigcup_{i=1}^\infty X_i$ is dense in X.

Proof of Lemma 3. Given $\delta > 0$, let $\{x_1, x_2, \ldots, x_k\} \subset \bigcup_1^\infty X_i$ be a δ-net of $f(A)$. Since A is finite-dimensional, there exists a cover $\{U_i\}_{i=1}^k$ of finite order such that $U_i \subset \{a \in A\colon d(f(a), x_i) < \delta\}$. Let

$$\lambda_i(a) = \operatorname{dist}(a, A \setminus U_i) / \max_{1 \leqslant j \leqslant k} \{\operatorname{dist}(a, X \setminus U_j)\}.$$

Write $\bar{f}_0(a) \equiv x_0$ for some $x_0 \in X$, and $\bar{f}_i(a) = c_{\lambda_i(a)}(\bar{f}_{i-1}(a), x_i)$ for $i = 1, 2, \ldots, k$. Note that for each $a \in A$, there exists an i such that $\lambda_i(a) = 1$; hence $\bar{f}_i(a) = x_i$. Consequently, if δ is small the map $\bar{f} = f_k$ is well defined and does not depend on x_0. It is easily seen that if δ is sufficiently small then $d(f, \bar{f}) < \varepsilon$.

The following theorem will be our main tool in applications. It summarizes the facts of Propositions 1, 2, and 3 concerning locally contractible σ-fd-compact groups.

THEOREM 1. *Let G be a σ-fd-compact metrizable topological group and let (G_i) be a tower of subgroups of G so that $\bigcup_1^\infty G_i$ is dense in G. Suppose there exists a homotopy (ϕ_t) contracting a neighborhood U of 1 to 1 in G such that for each m, $\phi_t(G_m \cap U) \subset G_n$ for some $n \geqslant m$ and all t. Moreover, suppose that for each m there exists for some $n > m$ a path $p_m = p\colon [0,1] \to G_n$ such that $p(0) = 1$, $p(1) \notin G_m$, and $\{p(t)(p(s))^{-1}\colon 0 \leqslant t, s \leqslant 1\} \cap G_m = \{1\}$. Then G is locally homeomorphic to l_2^f.*

PROOF. It is easy to see that $c_t(g_1, g_2) = g_2 \cdot (\phi_t(1))^{-1} \cdot \phi_t(g_2^{-1} \cdot g_1)$ defines a local equiconnecting map. To apply Proposition 3 it suffices to show that (G_i) is finitely expansive. We may assume that p is an arc. Let $u_m(g,t) = g \cdot p(t)$, $(g,t) \in G_m \times [0,1]$. If $u_m(g_1,t) = u_m(g_2,s)$, then $p(t)(p(s))^{-1} \in G_m$. Hence $p(t)(p(s))^{-1} = 1$ and consequently $t = s$ and $g_1 = g_2$. This shows that u_m is one-to-one. Since $p_m(0) = 1$, $u_m(g,0) = g$. Thus Proposition 3 ensures that $\bigcup G_i$ is a skeletoid. The existence of (ϕ_t) implies the local contractibility of G, so that Proposition 1 is applicable and G is an ANR. Proposition 2 yields the discrete approximation property (note that G is infinite-dimensional). The discrete approximation property in turn implies that every compact subset of G is a Z-set. Finally the strong universal property follows from [**6**, Proposition 2.2]. By a result of Mogilski [**15**], G is locally homeomorphic to l_2^f.

Note 1. If G is abelian and $q\colon [0,1] \to G_n$, $n \geqslant m$, is a path such that $q(0) \in G_m$, $q(1) \notin G_m$ and $\{q(t) - q(s)\colon 0 \leqslant t, s \leqslant 1\} \cap G_m = \{1\}$, then $p(t) = q(t) - q(0)$ may serve as a required path in Theorem 1.

Note 2. Assume additionally in Theorem 1 that G is a dense subgroup of a complete-metrizable topological group $\tilde{G}$. If $\tilde{G}$ is an ANR then there is an l_2-manifold M and an l_2^f-manifold N such that $(\tilde{G}, G)$ is homeomorphic to (M, N). This is a consequence of a result of [**8**] and the fact that G is a skeletoid in $\tilde{G}$ (see [**1**]).

4. The group of piecewise continuous measure preserving transformations of $[-1,1]$. Let l denote Lebesgue measure on $[-1,1]$. Consider the group H of all l-equivalence classes of l-measurable bijections of $[-1,1]$ that preserve the measure l. The group H will be endowed with the weak topology, i.e., the subbasis of neighborhoods of $g \in H$ is the collection consisting of $\{h \in H\colon l(gA \triangle hA) < \varepsilon\}$, where A is a measurable set. It is known [**10**] that H is a complete-metrizable separable topological group. It is conjectured that H is homeomorphic to l_2 (equivalently [**8**] H is an AR). Let G denote the subgroup of H consisting of all l-equivalence classes of piecewise continuous transformations. We always represent an element $g \in G$ by a left continuous transformation.

THEOREM 2. *The group G is homeomorphic to l_2^f. Moreover, the pairs (H, G) and (l_2, l_2^f) are homeomorphic provided H is an AR.*

First, we shall show that G is contractible. The contractibility of H was demonstrated by Keane [**13**]. However, Keane's method of induced transformations is not applicable in the case of G. We replace it by the Alexander trick. To describe it, we find it convenient to identify an element $g \in H$ with a measure preserving transformation $\bar{g}$ of the reals which is the extension of g by the identity off $[-1,1]$. Given $g \in H$ and $t \in (0,1]$, let $\phi_t(g) = t\bar{g}(t^{-1}x)$, $x \in [-1,1]$, and let $\phi_0(g) = 1 \in H$.

LEMMA 5. *The function* (ϕ_t) *and its restriction to G are homotopies contracting H and G to* 1, *respectively*.

PROOF. Clearly, if $g \in H$ then $\phi_t(g)$ is a measure preserving transformation, and if $g \in G$ then $\phi_t(g) \in G$. First we check the continuity of (ϕ_t) at $t = 0$. Let $(g_n, t_n) \to (g, 0)$ in $H \times [0,1]$. Since $\phi_{t_n}(g_n)$ is the identity outside $[-t_n, t_n]$, for every closed set $F \subset [-1,1]$ with $0 \notin F$ there exists n_0 such that $\phi_{t_n}(g_n)$ is the identity on F for all $n \geqslant n_0$. This yields the continuity of (ϕ_t) at $t = 0$. Next, let $(g_n, t_n) \to (g, t)$, $t \neq 0$. For any metrizable subset A of $[-1,1]$ we have

$$l\big(\phi_{t_n}(g)A \triangle \phi_t(g)A\big) \leqslant l\big(\phi_{t_n}(g_n)A \triangle \phi_t(g_n)A\big) + l\big(\phi_t(g_n)A \triangle \phi_t(g)A\big).$$

Since $g_n \to g$, we have $l(\bar{g}_n(t^{-1}A) \triangle \bar{g}(t^{-1}A)) \to 0$. The latter implies that

$$l\big(\phi_t(g_n)A \triangle \phi_t(g)A\big) = l\big(t\bar{g}_n(t^{-1}A) \triangle t\bar{g}(t^{-1}A)\big) \to 0.$$

Since each $\bar{g}_n$ is a measure preserving transformation and $t_n \to t$, we get

$$l\big(\bar{g}_n\big(t_n^{-1}A\big) \triangle \bar{g}_n(t^{-1}A)\big) = l\big(\bar{g}_n\big(t_n^{-1}A \triangle t^{-1}A\big)\big) = l\big(t_n^{-1}A \triangle t^{-1}A\big) \to 0.$$

We conclude that

$$l\big(\phi_{t_n}(g_n)A \triangle \phi_t(g)A\big) = l\big(t_n\bar{g}_n(t^{-1}A) \triangle t\bar{g}_n(t^{-1}A)\big) \to 0;$$

and continuity of (ϕ_t) follows.

LEMMA 6. *The group G is σ-fd-compact*.

PROOF. Let $K_i = \{g \in G$: is discontinuous at most at i points$\}$. We claim that K_i is a finite-dimensional compactum. Since $G = \bigcup_1^\infty K_i$, the lemma will follow. First we show the compactness of K_i. Pick a sequence $\{g_n\} \subset K_i$ and let g_n be discontinuous at $a_n^1 \leqslant a_n^2 \leqslant \cdots \leqslant a_n^i$. Passing to subsequences we may assume that $\{a_n^j\}$ and $\{g_n(a_n^j)\}$ converge for $j = 1, 2, \ldots, i$, and for each j the collection $\{g_n|[a_n^{j-1}, a_n^j)\}_{n=1}^\infty$ has a fixed orientation. We let $a_n^0 = -1$, a_n^{i+1} and we only consider such n that $[a_n^{j-1}, a_n^j) \neq \varnothing$. Let $\lim_n a_n^j = a_0^j$ and $\lim_n g_n(\hat{a}_n^j) = b_0^j$. Since, for every n, $a_n^j - a_n^{j-1} = |b_n^{j-1} - b_n^j|$, we also have $a_0^j - a_0^{j-1} = |b_0^{j-1} = b_0^j|$. There exists $g_0 \in K_n$ such that $g_0([a_0^{j-1}, a_0^j))$ is the segment joining b_0^{j-1} and b_0^j with the orientation of $\{g_n|[a_n^{j-1}, a_n^j)\}$. Since for every closed subsete $F \subset (a_0^{j-1}, a_0^j)$ we have $l(g_nF \triangle g_0F) \to 0$, we infer that $g_n \to g_0$.

Let $L_i = \{g \in G$: g is discontinuous at exactly i points$\}$. Fix $g \in G$ and let $a^1 < a^2 < \cdots < a^i$ be the discontinuity points of g. The sequence of orientations of $\{g|[a^{j-1}, a^j)\}$ represents an element of S, the space of $(i+1)$-sequences of $\{+,-\}$. Assign to g a sequence $v(g) = \{(a^j, g(a^j))\}_{j=0}^i \in (R \times R)^{i+1}$. Consider $v(g)$ as an element of the discrete union $Z = \bigoplus_{s \in S}(R \times R)^{i+1}$ in such a way that the sequence $s \in S$ of orientations of $\{g|[a^{j-1}, a^j)\}$ determines a summand in Z. We claim that v: $L_i \to Z$ is an embedding. The continuity of v follows from the fact we have just proved, that if $g_n \to g_0 \in L_i \subset K_i$, then there exists a subsequence $\{g_k\}$ such that $v(g_k) \to v(g_0)$. On the other hand if $v(g_n) \to v(g_0)$, then every subsequence of $\{g_n\}$ contains a convergent subsequence whose limit g belongs to K_i. Since $\lim v(g_n) = v(g_0)$, we conclude that $g \in L_i$ and $g = g_0$. This shows that v

is an embedding and that L_i is finite-dimensional. Finally K_i as a finite union of finite-dimensional sets is finite-dimensional itself.

PROOF OF THEOREM 2. Let $G_i = \{g \in G\colon\ g(x) = x \text{ for } |x| > 1 - 1/i\}$. By Lemma 5, $\bigcup_1^\infty G_i$ is dense in G. Moreover, the homotopy (ϕ_t) contracting G to 1 satisfies the condition $\phi_t(G_m) \subset G_m$ for all t. Pick $g \in G_{m+1} \setminus G_m$ such that $g(x) = x$ for $x < 1 - 1/m$ and write $p(t) = L^{-1}\phi_t(g)L$, where L is the linear map with $L(1 - 1/m) = -1$ and $L(1 - 1/(m+1)) = 1$. It is easily seen that all the assumptions of Theorem 1 are satisfied: consequently G is homeomorphic to l_2^f. Finally, by a result of Halmos [**10**], G is dense in H; so if H is an AR, then by Note 2, the pairs (H, G) and (l_2, l_2^f) are homeomorphic.

5. Groups spanned by linearly independent arcs. By a *function space* on $[0,1]$ we mean (cf. [**7**]) a metric linear space $(E, \|\cdot\|)$ of l-equivalence classes of real l-measurable functions on $[0,1]$ satisfying the following conditions:

(i) If $|f| \leqslant |g|$ a.e. and $g \in E$, then $f \in E$ and $\|\,|f|\,\| \leqslant \|\,|g|\,\|$,

(ii) $f(x) \equiv 1 \in E$,

(iii) Given $\{f_n\} \subset E$, $|f_n| \leqslant |f|$ a.e., $n = 1, 2, \ldots$, with $0 \leqslant f \in E$, we have $\|f_n\| \to 0$ iff $f_n \to 0$ in measure.

THEOREM 3. *Let $G = \mathrm{gr}\{\chi_{[0,t]}\colon 0 \leqslant t \leqslant 1\}$ be the group span of indicator functions $\chi_{[0,t]}$ in a function space E. Then G is homeomorphic to l_2^f. If E is complete and $\tilde{G}$ is the closure of G, then the pairs $(\tilde{G}, G)$ and (l_2, l_2^f) are homeomorphic.*

LEMMA 7. *The formula $\phi_t(g) = \chi_{[0,t]} \cdot g$ defines a homotopy contracting G to* 0.

PROOF. Since G consists of all finite sums $\sum n_i \chi_{I_i}$, where (n_i) are integers and (I_i) subintervals of $[0,1]$, $\phi_t(g) \in G$ for every $g \in G$. The continuity of (ϕ_t) was shown in [**7**].

We need the following fact whose proof, imitating an argument of [**1**, p. 282], can be easily provided.

LEMMA 8. *The group span of a linearly independent finite-dimensional compactum in an abelian topological group is σ-fd-compact.*

PROOF OF THEOREM 3. Let $G_i = \{\chi_{[0,1-1/i]}g\colon\ g \in G\}$. We have $\phi_t(G_m) \subset G_m$. If $g = \chi_{[1-1/m, 1-1/(m+1)]}$, then we set $p_m(t) = \phi_t(g)$. Theorem 1 is now applicable and G is homeomorphic to l_2^f. To get the second assertion note that $\tilde{G}$ consists of all integer-valued functions of E. By [**7**], $\tilde{G}$ is an AR, hence Note 2 is applicable and $(\tilde{G}, G)$ is homeomorphic to (l_2, l_2^f).

Let $\mathscr{L}$ denote *the measure algebra* of l-equivalence classes of l-measurable subsets of $[0,1]$. Endowed with $(A, B) \to A \triangle B$ as an addition and $d(A, B) = l(A \triangle B)$ as a metric, $\mathscr{L}$ is an abelian metrizable topological group.

THEOREM 4. *Let $\mathscr{L}_0 = \mathrm{gr}\{[0,t]\colon\ 0 \leqslant t \leqslant 1\} \subset \mathscr{L}$. Then the pairs $(\mathscr{L}, \mathscr{L}_0)$ and (l_2, l_2^f) are homeomorphic.*

PROOF. Identify $\mathscr{L}$ with the space of all indicator functions of $L_2[0,1]$; see [**1**, p. 200]. Since $\mathscr{L}_0$ is a closed subset of $\mathrm{gr}\{\chi_{[0,t]}\}_{0 \leqslant t \leqslant 1}$ in $L_2[0,1]$, it is σ-fd-compact. The formula of Lemma 7 defines a homotopy (ϕ_t) contracting $\mathscr{L}_0$ to 0. Letting

$\mathscr{L}_i = \{\chi_{[0,1-1/i]}g;\ g \in \mathscr{L}_0\}$ we see that $\phi_t(\mathscr{L}_m) \in \mathscr{L}_m$ for all t. Let p_m be as in the proof of Theorem 3. Now, the assertion follows from Theorem 1 and the fact **[1]** that $\mathscr{L}$ is homeomorphic to l_2.

Below we deal with a group spanned by an arc in a sequence space. Consider the map $\alpha\colon [0,1) \to l_2$ given by $\alpha(s) = (s, s^2, s^3, \dots)$. Note that

(a) $\alpha(0,1) \subset l_2$ is linearly independent.

(b) $|\sum n_i\alpha(ts_i)| \leqslant |t|\ |\sum n_i\alpha(s_i)|$ for every finite sum $\sum n_i\alpha(s_i)$ and all $t \in [0,1]$ ($|(x_i)| \leqslant |(y_i)|$ means $|x_i| \leqslant |y_i|$ for all i).

(c) $|\sum n_i\alpha(ts_i) - \sum n_i\alpha(t_0 s_i)| \leqslant |t - t_0||\sum n_i\alpha(s_i)|$ for every sum $\sum n_i\alpha(s_i)$ and all $t, t_0 \in [0,1]$.

THEOREM 5. *The group $G = \mathrm{gr}\{\alpha(s)\colon 0 \leqslant s < 1\} \subset l_2$ is homeomorphic to l_2^f. If $\overline{G}$ is an AR then $(\overline{G}, G)$ is homeomorphic to (l_2, l_2^f).*

LEMMA 9. *The formula $\phi_t(g) = \sum n_i\alpha(ts_i)$, $g = \sum n_i\alpha(s_i) \in G$, defines a homotopy contracting G to 0 (and naturally extends to a homotopy contracting $\overline{G}$).*

PROOF. We have $\|\phi_t(g) - \phi_t(g_0)\| \leqslant \|\phi_t(g - g_0)\| + \|\phi_t(g_0) - \phi_{t_0}(g_0)\|$. The two terms can be made as small as we wish due to (b) and (c), respectively, provided (g, t) is close to (g_0, t_0).

Let $\{g_n\}$ be a Cauchy sequence in G. By (b), $\|\phi_t(g_n) - \phi_t(g_m)\| \leqslant t\|g_n - g_m\|$. Consequently, each ϕ_t extends to a map $\tilde{\phi}_t\colon \overline{G} \to \overline{G}$. Using the properties of (b) and (c), it is routine to check that $(\tilde{\phi}_t)$ is a homotopy.

PROOF OF THEOREM 5. Let $G_i = \mathrm{gr}\,\alpha([0, 1 - 1/i])$. We have $\phi_t(G_m) \subset G_m$ for all t. Setting $p_m(t) = \phi_t(g)$, where $g = \alpha(1 - 1/(m+1))$, the last assumption of Theorem 1 is satisfied via Note 1. Since, by Lemma 8, G is σ-fd-compact, Theorem 1 and Note 2 are applicable, yielding the assertion.

The consideration of this section suggests the problem of the topological classification of additive subgroups of Banach spaces. In particular the following seems to be reasonable.

Question. Assume that G is a subgroup of a Banach space. Does $G \in LC^m$ imply $G \in LC^n$ for $1 \leqslant m \leqslant n \leqslant \infty$? Does $G \in LC^\infty$ (resp. $G \in LC$) imply that G is an ANR?

REFERENCES

1. C. Bessaga and A. Pełczynski, *Selected topics in infinite-dimensional topology*, PWN, Warsaw, 1975.

2. P. L. Bowers, *Dense embeddings of nowhere locally compact separable metric spaces*, preprint.

3. D. W. Curtis, *Some theorems and examples on local equiconnectedness and its specializations*, Fund. Math. **72** (1971), 101–113.

4. ______, *Sigma-compact, nowhere locally compact metric spaces can be densly imbedded in Hilbert space*, Topology Appl. **16** (1983), 253–257.

5. ______, *Boundary sets in the Hilbert cube*, Topology Appl. (to appear).

6. D. W. Curtis, T. Dobrowolski, and J. Mogilski, *Some applications of the topological classification of the sigma-compact spaces l_2^f and Σ*, Trans. Amer. Math. Soc. **284** (1984), 837–846.

7. T. Dobrowolski, *On extending mappings into nonlocally convex linear metric spaces*, Proc. Amer. Math. Soc. **93** (1985), 555–560.

8. T. Dobrowolski and H. Toruńczyk, *Separable complete ANRs admitting group structures are Hilbert manifolds*, Topology Appl. **12** (1981), 229–235.

9. S. Eilenberg and R. L. Wilder, *Uniform local connectedness and contractibility*, Amer. J. Math. **64** (1942), 613–622.

10. P. R. Halmos, *Lectures on ergodic theory*, Chelsea, New York, 1956.

11. W. E. Haver, *Locally contractible spaces that are absolute retracts*, Proc. Amer. Math. Soc. **40** (1973), 280–286.

12. K. H. Hofmann, *Homogeneous locally compact groups with compact boundary*, Trans. Amer. Math. Soc. **106** (1963), 52–63.

13. M. Keane, *Contractibility of the automorphism group of a nonatomic measure space*, Proc. Amer. Math. Soc. **26** (1970), 420–422.

14. J. L. Kelley, *General topology*, Van Nostrand, Princeton, N. J., 1955.

15. J. Mogilski, *Characterizing the topology of infinite-dimensional σ-compact manifolds*, Proc. Amer. Math. Soc. **92** (1984), 111–118.

16. H. Toruńczyk, *Concerning locally homotopy negligible sets and characterizations of l_2-manifolds*, Fund. Math. **101** (1978), 93–110.

17. ______, *Characterizing Hilbert space topology*, Fund. Math. **111** (1981), 247–262.

DEPARTMENT OF MATHEMATICS, TEXAS A & M UNIVERSITY, COLLEGE STATION, TEXAS 77843 (Current address)

INSTITUTE OF MATHEMATICS, WARSAW UNIVERSITY, 00 - 901 WARSAW PKiN, IX P, POLAND