

MAPS IN \mathbf{R}^n WITH FINITE-TO-ONE EXTENSIONS

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ABSTRACT. Suppose $f: X \rightarrow \mathbf{R}^n$ is a continuous function from a closed subset X of \mathbf{R}^n into \mathbf{R}^n . The Tietze Extension Theorem states that there is a continuous function $F: \mathbf{R}^n \rightarrow \mathbf{R}^n$ that extends f . Here we consider the question of when the extension F can be chosen with $F|_{\mathbf{R}^n - X}$ being finite-to-one. Not every map f has such an extension. If $f(X)$ is sufficiently nice, then there is such a finite-to-one extension. For example, it is shown that if $f: X \rightarrow \mathbf{R}^n$ is a map and $f(X) \subset \mathbf{R}^{n-1} \times \{0\}$ then there is a continuous extension $F: \mathbf{R}^n \rightarrow \mathbf{R}^n$ such that $F|_{\mathbf{R}^n - X}$ is finite-to-one. On the other hand, if X is nowhere dense and $f(X)$ contains an open set, then there definitely is not such a finite-to-one extension. Other examples and theorems show that the finite-to-one extendability of a map $f: X \rightarrow \mathbf{R}^n$ is not necessarily a function of the topology of $f(X)$, but may depend on its embedding or on the map f .

1. Introduction. Suppose $f: X \rightarrow \mathbf{R}^n$ is a continuous function from a closed subset X of \mathbf{R}^n into \mathbf{R}^n . The Tietze Extension Theorem states that there is a continuous function $F: \mathbf{R}^n \rightarrow \mathbf{R}^n$ that extends f . Here we consider the question of when the extension F can be chosen with $F|_{\mathbf{R}^n - X}$ being finite-to-one. (Throughout this paper, every function mentioned is continuous. In particular, maps and extensions are all continuous functions.)

First we show in Theorem 2.1 that for any closed set X in \mathbf{R}^n and any map $f: X \rightarrow \mathbf{R}^n$ there is a continuous extension $F: \mathbf{R}^n \rightarrow \mathbf{R}^n$ such that $F|_{\mathbf{R}^n - X}$ is countable-to-one. If $f(X)$ lies in a nice subspace of \mathbf{R}^n such as $\mathbf{R}^{n-1} \times \{0\}$, then an extension $F: \mathbf{R}^n \rightarrow \mathbf{R}^n$ can be chosen so that $F|_{\mathbf{R}^n - X}$ is finite-to-one.

Not all maps f have such finite-to-one extensions. Example 3.1 is an example of a compact subset X of \mathbf{R}^2 and a reembedding $f: X \rightarrow \mathbf{R}^2$ such that no extension $F: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is finite-to-one. In this example, X contains simple closed curves whose separation properties are used in establishing the properties of the example. Other examples in this paper depend on other properties for their proofs.

In Theorem 3.2 it is proved that if f is a map from a nowhere dense closed subset X of \mathbf{R}^n into \mathbf{R}^n , then no extension $F: \mathbf{R}^n \rightarrow \mathbf{R}^n$ of f is finite-to-one on $\mathbf{R}^n - X$.

In the final section, we investigate the finite-to-one extendability of functions f from the standard Cantor set C into \mathbf{R}^2 . From the result mentioned above it follows that no map $f: C \rightarrow \mathbf{R}^2$ where $f(C)$ contains an open set has an extension $F: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ with $F|_{\mathbf{R}^2 - C}$ being finite-to-one. This same phenomenon can occur

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even if $f(C)$ is 1-dimensional. Example 4.2 shows that if f is a map from C onto $[0, 1] \times C$ in \mathbf{R}^2 , then no extension $F: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ of f is finite-to-one on $\mathbf{R}^2 - C$.

In the above-mentioned theorems, the topological type of the image $f(C)$ determined whether or not f has the finite-to-one extendability property.

The finite-to-one extendability of a map $f: C \rightarrow \mathbf{R}^n$ is not necessarily determined by the topological type of $f(C)$. In Theorem 2.3, it is shown that if $f(C)$ is a subset of $[0, 1] \times (\bigcup_{n=1}^{\infty} \{1/n\} \cup \{0\})$ in \mathbf{R}^2 , then there is a continuous extension $F: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ such that $F|_{\mathbf{R}^2 - C}$ is finite-to-one. On the other hand, there is a map (finite-to-one in fact) f from C to $[0, 1] \times (\bigcup \{\pm 1/n\} \cup \{0\})$ with no continuous finite-to-one extension. Since $[0, 1] \times (\bigcup \{1/n\} \cup \{0\})$ is homeomorphic to $[0, 1] \times (\bigcup \{\pm 1/n\} \cup \{0\})$, we see that the finite-to-one extendability of a map is not necessarily determined by the topological type of the image.

An unanswered question is whether there is a compact set X of \mathbf{R}^2 such that for every map f from C into X , f admits a continuous extension F that is finite-to-one on $\mathbf{R}^2 - C$, and yet there is a reembedding $h: X \rightarrow \mathbf{R}^2$ such that for every map f from C onto $h(X)$, there is no extension $F: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ that is finite-to-one on $\mathbf{R}^2 - C$.

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2. Images in \mathbf{R}^n allowing finite-to-one extensions. In this section we show first that every map f from a closed subset X of \mathbf{R}^n into \mathbf{R}^n admits an extension $F: \mathbf{R}^n \rightarrow \mathbf{R}^n$ that is countable-to-one on $\mathbf{R}^n - X$. Later theorems show that an extension $F: \mathbf{R}^n \rightarrow \mathbf{R}^n$ above can be chosen to be finite-to-one on $\mathbf{R}^n - X$ if the image $f(X)$ is sufficiently nice.

THEOREM 2.1. *Let $f: X \rightarrow \mathbf{R}^n$ be a continuous function where X is a closed subset of \mathbf{R}^n . Then there is a continuous extension $F: \mathbf{R}^n \rightarrow \mathbf{R}^n$ of f such that $F|_{\mathbf{R}^n - X}$ is countable-to-one, i.e., for each $x \in \mathbf{R}^n - X$, $|F^{-1}(F(x))| \leq \omega_0$.*

PROOF. Let T be a triangulation of $\mathbf{R}^n - X$ such that for any $\varepsilon > 0$ any simplex σ of T with $d(\sigma, X) < \varepsilon$ has diameter less than ε .

For each vertex v_i of T choose a point x_i of X such that $d(v_i, x_i) = d(v_i, X)$. Define $F(v_i)$ to be any point within $d(v_i, x_i)$ of $f(x_i)$. Choose the points $\{F(v_i)\}$ in general position. Define F on $\mathbf{R}^n - X$ by extending linearly over each simplex of T . Since there are only countably many simplexes of T and each one is embedded by F , $F|_{\mathbf{R}^n - X}$ is countable-to-one. \square

THEOREM 2.2. *Let X be a closed subset of \mathbf{R}^n and $f: X \rightarrow \mathbf{R}^{n-1} \times \{0\}$ be a continuous function. Then there is a continuous extension $F: \mathbf{R}^n \rightarrow \mathbf{R}^n$ of f such that $F|_{\mathbf{R}^n - X}$ is finite-to-one.*

PROOF. Let T be a triangulation of $\mathbf{R}^n - X$ such that for each simplex σ of T ,

(i) for each $\varepsilon > 0$, if $d(\sigma, X) < \varepsilon$, then $\text{diam } \sigma < \varepsilon$; and

(ii) if for any point p of σ , $d(p, \text{origin}) > N$, then $\text{diam } \sigma < 1/N$.

Thus simplices of T close to X are small and simplices far from the origin are small.

The extension F of f will be linear on each simplex of T . For each vertex v of T choose a point x of X with $d(v, x) = d(v, X)$. Define $F(v)$ to be a point in \mathbf{R}_+^n ,

(i.e., $\mathbf{R}^{n-1} \times (0, \infty)$) such that

$$d(F(v), f(x)) < \min\{d(v, X), \text{diam } \sigma \text{ where } \sigma \text{ is any } n\text{-simplex of } T \text{ containing } v\}.$$

Choose the $F(v)$'s in general position. The map $F: \mathbf{R}^n \rightarrow \mathbf{R}^n$ is defined to be f on X and on each simplex of T , F is the linear extension determined by the $F(v)$'s chosen above.

The map F is a continuous extension of f . We now show that $F|_{\mathbf{R}^n - X}$ is finite-to-one. Note first that $F^{-1}(\mathbf{R}^{n-1} \times \{0\}) = X$ since every point of $\mathbf{R}^n - X$ lies in a simplex of T each of whose vertices is mapped by F into \mathbf{R}^n_+ . Consider, therefore, a point $p \in \mathbf{R}^n_+$. If $d(p, \mathbf{R}^{n-1} \times \{0\}) > \epsilon$ and $z \in F^{-1}(p)$, then by having chosen the $F(v)$'s such that $d(F(v), f(x)) < \text{diam } \sigma$ where σ is any n -simplex of T containing v , we know that the diameter of any n -simplex containing z must be greater than ϵ . However, there are only finitely many simplices of T of diameter greater than ϵ and each is embedded by F . Therefore $F^{-1}(p)$ is finite. \square

(Ric Ancel and the author proved that $\mathbf{R}^{n-1} \times \{0\}$ could not be replaced in Theorem 2.2 by an arbitrary embedding of \mathbf{R}^{n-1} in \mathbf{R}^n . In particular, if $f: \mathbf{R}^2 \times \{0\} \rightarrow \mathbf{R}^3$ is an embedding of \mathbf{R}^2 such that $f(\mathbf{R}^2 \times \{0\})$ has dense sets of Fox-Artin feelers on each side, then no extension of f is finite-to-one.)

We next consider one more image with this finite-to-one extension property. This space is of interest in conjunction with Example 4.3.

Once again, the proof of this theorem is a minor modification of the proof of Theorem 2.2.

THEOREM 2.3. *Let X be a closed subset of \mathbf{R}^n and $f: X \rightarrow \mathbf{R}^{n-1} \times (\bigcup_{n=1}^{\infty} \{1/n\}) \cup \{0\}$ be a continuous function. Then there is a continuous extension $F: \mathbf{R}^n \rightarrow \mathbf{R}^n$ of f such that $F|_{\mathbf{R}^n - X}$ is finite-to-one.*

PROOF. The construction of T and F is identical to that in the proof of Theorem 2.2 except for the conditions on the choice of the images of the vertices $F(v)$. Here for each vertex v we choose $F(v)$ such that

- (i) $d(F(v), f(x)) < \min\{d(v, X), \text{diam } \sigma \text{ where } \sigma \text{ is any } n\text{-simplex of } T \text{ containing } v\}$;
- (ii) the n th coordinate of $F(v)$ is greater than the n -coordinate of $f(x)$;
- (iii) the $F(v)$'s are in general position.

With this slight modification in (ii) above, one proves as before that $F|_{\mathbf{R}^n - X}$ is finite-to-one. \square

3. Images not allowing finite-to-one extensions. In this section we construct examples of continuous functions $f: X \rightarrow \mathbf{R}^n$ where X is a closed subset of \mathbf{R}^n such that f does not extend to a map $F: \mathbf{R}^n \rightarrow \mathbf{R}^n$ with $F|_{\mathbf{R}^n - X}$ finite-to-one.

EXAMPLE 3.1. This example is a continuous function $f: X \rightarrow \mathbf{R}^2$ where X is a compact subset of \mathbf{R}^2 such that f is finite-to-one (in fact, one-to-one) and yet f has no finite-to-one extension.

Let $X = \{(0, 0)\} \cup \bigcup_{i=1}^{\infty} S_i$ where S_i is a circle of radius $1/2^{i+2}$ centered at $(1/2^i, 0)$. Therefore, X is a compact subset of \mathbf{R}^2 equal to the union of countably

many simple closed curves bounding disjoint disks together with a limit point. Let $f: X \rightarrow \mathbf{R}^2$ be defined by $f((0,0)) = (0,0)$ and $f(S_i) =$ the circle of radius $1/2^i$ centered at $(0,0)$.

Let $F: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be any extension of f . For each disk D_i bounded by S_i , there is a point x_i in $\text{Int } D_i$ such that $F(x_i) = (0,0)$. Hence $F|_{\mathbf{R}^2 - X}$ is not finite-to-one. \square

Example 3.1 made use of separation properties of X . The lemma and theorem below show that space-filling maps from a nowhere dense subset X of \mathbf{R}^n cannot have finite-to-one extensions. Lemma 3.1 is a special case of the well-known theorem that closed light maps cannot lower dimension [HW, Theorem VI 7, p. 91].

LEMMA 3.1. *Let $f: D^n \rightarrow \mathbf{R}^n$ be a map with $f^{-1}(f(x))$ totally disconnected for each $x \in D^n$. Then $f(D^n)$ contains an open set. In fact, $f(D^n)$ has a dense open subset.* \square

THEOREM 3.2. *Let X be a nowhere dense closed set in \mathbf{R}^n and $f: X \rightarrow \mathbf{R}^n$ be a map such that $f(X)$ contains an open set. Then no extension $F: \mathbf{R}^n \rightarrow \mathbf{R}^n$ of f is finite-to-one on $\mathbf{R}^n - X$.*

PROOF. Suppose X and f are given as above and there is an extension F which is finite-to-one off X . Let U be an open set in $f(X)$ by hypothesis. Since F is continuous, $F^{-1}(U)$ is open. Since X is nowhere dense, there is a closed disk D_1 in $F^{-1}(U) - X$. By Lemma 3.1, there is an open set U_1 in $F(D_1)$. Since $U_1 \subset U$, $F^{-1}(U_1) \cap X \neq \emptyset$. Therefore, there is a small disk D_2 in $F^{-1}(U_1) - X - D_1$ since $d(D_1, X) > 0$.

Carry on defining disjoint disks D_i in $\mathbf{R}^n - X$ such that $F(D_{i+1}) \subset F(D_i)$. Then $\bigcap_{i=1}^{\infty} F(D_i) \neq \emptyset$. But for any point $x \in \bigcap F(D_i)$, x has infinitely many preimages in $\mathbf{R}^n - X$, since for each i , $F^{-1}(x) \cap D_i \neq \emptyset$. \square

4. Extendability of maps from the Cantor set. We can construct a finite-to-one map from the Cantor set C onto the unit square in \mathbf{R}^2 which by Theorem 3.2 does not extend to a finite-to-one map.

EXAMPLE 4.1. Example 4.1 is a well-known finite-to-one map f of the Cantor set C into \mathbf{R}^2 which Theorem 3.2 implies cannot be extended to a finite-to-one map $F: \mathbf{R}^2 \rightarrow \mathbf{R}^2$. Let C be the standard Cantor set whose points are given ternary representation using 0's and 2's only.

Define $f: C \rightarrow [0,1] \times [0,1]$, an onto map by: $f(.a_1a_2a_3 \cdots) = (.\hat{a}_1\hat{a}_3\hat{a}_5 \cdots, .\hat{a}_2\hat{a}_4\hat{a}_6 \cdots)$ where

$$\hat{a}_i = \begin{cases} 0 & \text{if } a_i = 0, \\ 1 & \text{if } a_i = 2 \end{cases}$$

and where the image points are interpreted as binary representations of points in $[0,1] \times [0,1]$.

Then f is 4-to-1 at most and $f(C) = [0,1] \times [0,1]$. By Theorem 3.2 no extension of f is finite-to-one. \square

There are 1-dimensional spaces which share the property with the square that Cantor set maps onto them do not have finite-to-one extensions. That is, Example

4.2 is a 1-dimensional space Y in \mathbb{R}^2 with the property that if $f: C \rightarrow Y$ is any onto map, then no extension $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of f is finite-to-one on $\mathbb{R}^2 - C$. The technical lemma below is used in establishing the properties of this example.

LEMMA 4.1. *If $f: C \rightarrow \mathbb{R}^2$ is a map from the Cantor set C into \mathbb{R}^2 satisfying the following properties, then no extension $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of f is finite-to-one on $\mathbb{R}^2 - C$. The properties of f are that there is a 1-dimensional continuum Y in $f(C)$ such that for every subcontinuum Z of Y there is a connected open set U in \mathbb{R}^2 with $U - Z$ not connected and a point $p \in C$ such that $f(p) \in (U \cap Z)$ and there are two sequences of points, $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ in C , each converging to p such that for each i , $f(x_i)$ and $f(y_i)$ are in U and are separated in U by Z .*

PROOF. As can be expected from the complexity of the statement of this lemma, the proof is straightforward.

Let f be a map satisfying the hypotheses and suppose F is an extension that is finite-to-one on $\mathbb{R}^2 - C$. We will produce a sequence of disjoint disks $\{D_i\}_{i=1}^\infty$ in $\mathbb{R}^2 - C$ and a sequence of nested subcontinua Z_i in X such that $Z_i \subset f(D_i)$ for every i . This construction demonstrates that F is not finite-to-one on $\mathbb{R}^2 - C$ since for every point $q \in \bigcap Z_i$, $f^{-1}(q)$ contains at least one preimage in each of the disjoint D_i 's.

The construction proceeds inductively as follows: Let $Z_1 = X$. Let U_1 be any connected open set in \mathbb{R}^2 such that $U_1 - Z_1$ is not connected. Find the hypothesized point p of C . By the convergence hypothesis on the sequences $\{x_i\}$ and $\{y_i\}$ and the continuity of F_1 choose an i and an arc α between x_i and y_i such that α intersects C only at its endpoints and such that $f(\alpha) \subset U_1$. Let D_1 be a disk in $\mathbb{R}^2 - C$ obtained by taking a regular neighborhood of a closed subarc of α such that $F(D_1) \subset U_1$ and for some two points, x'_i, y'_i of D_1 , $F(x'_i)$ is separated from $F(y'_i)$ in U_1 by Z_1 . Since $F^{-1}(Z_1)$ separates x'_i and y'_i in D_1 , $F^{-1}(Z_1)$ contains a 1-dimensional continuum. The image of that continuum (by Lemma 3.1) is a 1-dimensional subcontinuum Z_2 of Z_1 . We proceed inductively by choosing an open, connected set U_2 that is separated by Z_2 and repeating the construction, being careful to choose D_2 disjoint from D_1 as well as C . Repeating this process infinitely completes the proof of Lemma 4.1. \square

EXAMPLE 4.2. Let C be the Cantor set and let $f: C \rightarrow ([0, 1] \times C)$ be any onto map. Then no continuous extension $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of f is finite-to-one on $\mathbb{R}^2 - C$.

PROOF. We claim that there must be a point c in C such that $[0, 1] \times \{c\}$ satisfies the hypotheses for Y of Lemma 4.1.

Let us call a point (t, c) in $[0, 1] \times C$ a *good point* if and only if there is a point p in $f^{-1}(t, c)$ and two sequences of points $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ in C each converging to p such that for each i , $f(x_i)$ has a y -coordinate larger than c while $f(y_i)$ has a y -coordinate less than c . Other points in $[0, 1] \times C$ are *bad points*. We will show that there is a c in C such that every point of $[0, 1] \times C$ is a good point.

Suppose to the contrary, that for each c in C , there is a point (t, c) in $[0, 1] \times C$ that is a bad point. For each point c in C , choose one point c' in C such that $f(c') = (t, c)$, a bad point. Let $C_0 = \{c'\}$.

We will use the definition of bad points to divide C_0 into c 's with neighborhoods mapped high, H , and neighborhoods mapped low, L . Some c 's may have neighborhoods mapped into a horizontal line. These points are in both H and L .

Let $\{B_i\}$ be a countable basis for C . Let $H = \{c' \in C_0 \mid \text{there is a basic open set } B_{i(c')} \text{ in } C \text{ such that } c' \in B_{i(c')} \text{ and for each point } x \text{ of } B_{i(c')}, \text{ the } y\text{-coordinate of } f(x) \text{ is greater than or equal to the } y\text{-coordinate of } f(c')\}$. Let L be defined identically to H except replace "greater" by "less".

One of H or L is uncountable. We suppose H is. Therefore, there is a basic open set B_i such that $B_i = B_{i(c'_1)} = B_{i(c'_2)}$ for two points c'_1 and c'_2 in H . However, the y -coordinates of $f(c'_1)$ and $f(c'_2)$ are different. This contradicts the property about $B_{i(c')}$ in the definition of H .

We have proved, then, that there is a c in C such that every point in $[0, 1] \times C$ is a good point. But then $[0, 1] \times C$ satisfies the hypotheses of Y in Lemma 4.2. Therefore, the properties of this example are established. \square

The square and the product of the Cantor set with an interval have the property that *no* map of the Cantor set C onto either of these spaces can be extended to a map $F: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ that is finite-to-one off C . Earlier in Theorems 2.2 and 2.3 we saw spaces where *every* map from C into them has an extension that is finite-to-one off C .

Our next example shows that this all or none property with regard to extensions is not present for every image of C .

EXAMPLE 4.3. Let $X = [0, 1] \times (\bigcup_{n=1}^{\infty} \{\pm 1/n\} \cup \{0\})$. We will consider two finite-to-one maps of the Cantor set onto X . One has a finite-to-one extension; the other does not.

The extendable map $f_1: C \rightarrow X$ can be chosen to be any finite-to-one map defined as follows: Let $C = C_1 \cup C_2$ where C_1 and C_2 are disjoint sub-Cantor sets of C . Choose a finite-to-one map taking C_1 onto $[0, 1] \times (\bigcup_{n=1}^{\infty} \{1/n\} \cup \{0\})$ and another finite-to-one map taking C_2 onto $[0, 1] \times (\bigcup_{n=1}^{\infty} \{-1/n\} \cup \{0\})$. The union of the two maps is a finite-to-one map f_1 . The map f_1 can be extended to a finite-to-one map $F_1: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ with the aid of Theorem 2.3.

To describe the nonextendable map, consider the domain Cantor set to be $C \times (\bigcup_{n=1}^{\infty} \{\pm 1/n\} \cup \{0\})$. The map f_2 is defined by taking each sub-Cantor set $C \times \{t\}$ on $[0, 1] \times \{t\}$ by the standard two-to-one map. This finite-to-one map f_2 has no finite-to-one extension because the hypotheses of Lemma 4.1 are satisfied where the 1-dimensional subcontinuum mentioned in Lemma 4.1 is $[0, 1] \times \{0\}$. \square

Note that the space X in Example 4.3, namely, $[0, 1] \times (\bigcup_{n=1}^{\infty} \{\pm 1/n\} \cup \{0\})$ is homeomorphic to $Y = [0, 1] \times (\bigcup_{n=1}^{\infty} \{1/n\} \cup \{0\})$. Every map from C into Y has the finite-to-one extension property by Theorem 2.3 while Example 4.3 shows that not every map to X has that property. We see then that finite-to-one extendability of a map from C is not completely determined by the homeomorphism type of the image. This phenomenon raises the following question:

Question. Is there a compact set X in \mathbf{R}^n such that for every map $f: C \rightarrow X$, there is an extension $F: \mathbf{R}^n \rightarrow \mathbf{R}^n$ such that $F|_{\mathbf{R}^n - C}$ is finite-to-one; and yet, there is a reembedding $h(X)$ of X into \mathbf{R}^n such that no onto map $f: C \rightarrow h(X)$ has an extension $F: \mathbf{R}^n \rightarrow \mathbf{R}^n$ that is finite-to-one on $\mathbf{R}^n - C$?

In all the theorems and examples in this paper, the ability or inability to find a desired extension did not depend on whether the original map was finite-to-one. This raises the following question:

Question. Is there a compact set X in \mathbf{R}^n such that every finite-to-one map $f: C \rightarrow X$ has a finite-to-one extension $F: \mathbf{R}^n \rightarrow \mathbf{R}^n$ and yet there is a map $g: C \rightarrow X$ such that no extension $G: \mathbf{R}^n \rightarrow \mathbf{R}^n$ of g is finite-to-one on $\mathbf{R}^n - C$?

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