CONCERNING CONTINUITY APART FROM A MEAGER SET

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ABSTRACT. Given a σ-ideal \( \mathcal{J} \) of subsets of a space \( X \), mappings \( f: X \to Y \) are investigated, such that \( f \mid X_0 \) is continuous for some closed \( X_0 \subseteq X \) with \( X \setminus X_0 \in \mathcal{J} \).

1. Introduction. Throughout the paper, \( X \) and \( Y \) are topological spaces, and \( \mathcal{J} \) is a σ-ideal of subsets of \( X \).

The following theorem was first proved by Kuratowski in 1930. (See [K1] and [KM, p. 408]; also [EFK] for a closely related result.)

If \( Y \) is second countable, then the statement

(a) \( f \mid X_0 \) is continuous for some \( X_0 \subseteq X \) with \( X \setminus X_0 \in \mathcal{J} \) is equivalent to

(b) for every open \( V \subseteq Y \), \( f^{-1}(V) = (U \setminus A) \cup B \), where \( U \) is open in \( X \), \( A \in \mathcal{J} \), and \( B \in \mathcal{J} \).

We are interested in characterizing mappings that admit continuous restriction to a set \( X_0 \subseteq X \) with an extra property in addition to \( X \setminus X_0 \in \mathcal{J} \). The present paper is to investigate the case of a closed \( X_0 \).

2. The theorem. It is an easy observation that if

(i) \( f \mid X_0 \) is continuous for some closed \( X_0 \subseteq X \) with \( X \setminus X_0 \in \mathcal{J} \),

then

(ii) for every open \( V \subseteq Y \), \( f^{-1}(V) = U \setminus A \) with \( U \) open in \( X \) and \( A \in \mathcal{J} \).

Indeed, let \( X_0 \) be as in (i) and take any closed \( F \subseteq Y \). For (ii) to hold true, \( f^{-1}(F) \) should have a form of \( E \setminus A \) with \( E \subseteq X \) closed and \( A \in \mathcal{J} \). Since

\[
 f^{-1}(F) = \left[ f^{-1}(F) \cap X_0 \right] \cup \left[ f^{-1}(F) \cap (X \setminus X_0) \right] 
\]

\[
 = \left( f \mid X_0 \right)^{-1}(F) \cup \left( f^{-1}(F) \cap (X \setminus X_0) \right) ,
\]

this is indeed the case.

We are now going to describe circumstances where (i) and (ii) are actually equivalent. To make some statements shorter, let us agree to say that \( f \) is \( \mathcal{J} \)-continuous whenever (ii) is satisfied.

THEOREM. Let \( Y \) be a regular space and assume that either

(I) \( Y \) is second countable, or

(II) \( X \) is hereditarily Lindelöf, or

(III) \( \mathcal{J} \) consists of all meager sets in \( X \).

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Let \( f: X \to Y \) be \( \mathcal{I} \)-continuous. Then \( f | X_0 \) is continuous for some closed \( X_0 \subset X \) with \( X \setminus X_0 \in \mathcal{I} \).

3. **Examples.** Before a proof, let us look at two examples showing that the hypotheses of the theorem are to some extent indispensable; namely that
in (I), "second countable" cannot be weakened to "Lindelöf",
in (II), "hereditarily Lindelöf" cannot be weakened to "Lindelöf",
in (III), \( \mathcal{I} \) cannot be just any \( \sigma \)-ideal, and regularity of \( Y \) is essential in all three versions.

A. Let \( X = Y = \mathbb{R} \) with topologies as follows. All points except for 0 are isolated in both spaces, while neighborhoods of 0 in \( X \) (respectively, in \( Y \)) are complements of finite (respectively, countable) sets. These spaces \( X \) and \( Y \) are known to have nice topological properties: \( X \) is compact Hausdorff, \( Y \) is regular Hausdorff and Lindelöf.

Let \( \mathcal{I} \) be the \( \sigma \)-ideal of all countable subsets of \( X \). Although the identity function \( f: X \to Y \) is obviously \( \mathcal{I} \)-continuous, its restriction to any closed, co-countable set \( X_0 \subset X \) is discontinuous at 0.

B. Now consider \( X = \mathbb{R} \) with its usual topology. Let \( Y = \mathbb{R} \) have a richer topology consisting of all sets \( U \setminus A \), where \( U \) is in the usual topology of \( \mathbb{R} \) and \( A \) is a subset of \( Z = \{ 1/n : n \in \mathbb{N} \} \). Thus \( Y \) is Hausdorff and second countable.

Let \( \mathcal{I} \) be the \( \sigma \)-ideal of, say, all meager sets in \( X \) (or all sets of Lebesgue measure 0, or all countable sets). Since \( Z \in \mathcal{I} \), the identity function \( f: X \to Y \) is \( \mathcal{I} \)-continuous. On the other hand, the only closed set in \( X \) whose complement is a member of \( \mathcal{I} \) is \( X \) itself. Therefore, the only restriction of \( f \) to be considered is \( f \) itself, and that is not continuous.

4. **Proof of the theorem.** In case (I), let \( \{ V_n : n \in \mathbb{N} \} \) be a base for the topology of \( Y \). By the \( \mathcal{I} \)-continuity of \( f \), to each \( V_n \) there correspond an open set \( U_n \subset X \) and a set \( A_n \in \mathcal{I} \) so that
(1) \( f^{-1}(V_n) = U_n \setminus A_n \).

Put
(2) \( I = \bigcup \{ A_n : n \in \mathbb{N} \} \)
and
(3) \( X_0 = X \setminus \text{int } I \).

Of course, \( X_0 \) is closed in \( X \) and \( X \setminus X_0 \subset I \in \mathcal{I} \). To show that \( f | X_0 \) is continuous, take any \( x \in X_0 \) and any neighborhood \( V \) of \( f(x) \) in \( Y \). Since \( Y \) is regular, there is an \( n \in \mathbb{N} \) such that \( f(x) \in V_n \) and
(4) \( \text{cl } V_n \subset V \).

We will show that the corresponding neighborhood \( U_n \) of \( x \) in \( X \), defined by (1), satisfies \( f(U_n \cap X_0) \subset V \); this will prove that \( f | X_0 \) is continuous at \( x \).

Suppose that, on the contrary, \( f(x') \notin V \) for some \( x' \in U_n \cap X_0 \). Then, by (4), \( f(x') \in Y \setminus \text{cl } V_n \) and so, by the definition of a base, there is a \( V_m \) with \( f(x') \in V_m \) and
(5) \( V_m \cap V_n = \emptyset \).
The corresponding set $U_m$ contains $x'$; therefore
\[ x' \in U_m \cap U_n \cap X_0. \tag{6} \]

On the other hand, (1) and (2) yield $U_n \subset f^{-1}(V_n) \cup I$ and likewise, $U_m \subset f^{-1}(V_m) \cup I$. Hence, in view of (4),
\[ U_n \cap U_m \subset f^{-1}(V_n \cap V_m) \cup I = I \]
and consequently, $U_n \cap U_m \subset \operatorname{int} I$. Now (3) gives $U_n \cap U_m \cap X_0 = \emptyset$, a contradiction to (6).

Cases (II) and (III) can be given a common proof. But first, let us recall a theorem known as Banach Category Theorem. (See [B] and [O, p. 62].)

*If a subset $Z \subset X$ is locally meager then $Z$ is meager.*

An immediate corollary to the Banach Category Theorem is the following:

*The union of all subsets of $X$ that are open and meager at the same time is meager.*

In other words, in case (III) of the theorem, the following condition \((\ast)\) is satisfied:

\[ (\ast) \quad \text{The union of all members of $I$ that are open in $X$ is a member of $I$.} \]

The above holds in case (II) as well, since the union of all such sets (in a hereditarily Lindelöf space) coincides with the union of just countably many of them, which is in $\mathcal{F}$ by the definition of a $\sigma$-ideal. Thus \((\ast)\) is a property that (II) and (III) have in common.

Let us say that $f$ is *pointwise $\mathcal{F}$-continuous* if, for every $x \in X$ and for every neighborhood $V$ of $f(x)$ in $Y$, there exists a neighborhood $U$ of $x$ in $X$ such that $U \setminus f^{-1}(V) \in \mathcal{F}$.

Obviously, every $\mathcal{F}$-continuous mapping is pointwise $\mathcal{K}$-continuous. Therefore, to complete a proof of the theorem, it will suffice to prove the following lemma.

**Lemma.** Let $Y$ be a regular space and let $f$ be pointwise $\mathcal{F}$-continuous. Put
\[ X_0 = X \setminus \bigcup \{ G : G \text{ is open in } X \text{ and } G \in \mathcal{F} \}. \]

Then $f|_{X_0}$ is continuous.

**Proof.** Consider any point $x \in X_0$ and any neighborhood $V$ of $f(x)$ in $Y$. We need a neighborhood $U$ of $x$ in $X$ such that $f(U \cap X_0) \subset V$.

First, using regularity of $Y$, we can select an open set $W$ containing $f(x)$ with $\operatorname{cl} W \subset V$. Since $f$ is pointwise $\mathcal{F}$-continuous, a required neighborhood $U$ can now be chosen so that $U \setminus f^{-1}(W) \in \mathcal{F}$.

To show that $f(U \cap X_0) \subset V$, take any $x' \in U \cap X_0$; since $\operatorname{cl} W \subset V$, it will be enough to prove that $f(x') \in \operatorname{cl} W$.

Suppose that, on the contrary, there is a neighborhood $W'$ of $f(x')$ such that $W' \cap W' = \emptyset$. By the pointwise $\mathcal{F}$-continuity of $f$, we have $U' \setminus f^{-1}(W') \in \mathcal{F}$ for some neighborhood $U'$ of $x'$. Hence
\[ U \cap U' = (U \cap U') \setminus f^{-1}(W \cap W') \subset [U \setminus f^{-1}(W)] \cup [U' \setminus f^{-1}(W')] \in \mathcal{F}, \]
and therefore $U \cap U' \subset X \setminus X_0$, by the definition of $X_0$.

We have reached a contradiction, because on the other hand, $x' \in U \cap U' \cap X_0$.  

5. $\mathcal{F}$-continuity versus pointwise $\mathcal{F}$-continuity. As noted, every $\mathcal{F}$-continuous mapping is pointwise $\mathcal{F}$-continuous. The converse is not true in general.

For example, let $X = X_1 \times X_2$, where $X_1$ is a discrete uncountable space and $X_2 = \{0, 1, 1/2, 1/3, \ldots\}$ with the usual topology. Let $\mathcal{F}$ be the $\sigma$-ideal of all countable subsets of $X$. Since the space $X$ is locally countable, any mapping defined on it will automatically be pointwise $\mathcal{F}$-continuous. Not necessarily $\mathcal{F}$-continuous, though, as witnessed by the characteristic function of the set $X_1 \times \{0\}$. Notice that (*) is not satisfied in this example.

For another example, let $X$ and $\mathcal{F}$ be same as above, except that $X_2$ is now $\mathbb{R}$ with the usual topology. Let $Y$ be as in the second example of §3. It is easy to see that the mapping $f(x_1, x_2) = x_2$ is pointwise $\mathcal{F}$-continuous but not $\mathcal{F}$-continuous, although (*) is now satisfied in $X$.

However, given both (*) in $X$ and regularity of $Y$, every pointwise $\mathcal{F}$-continuous mapping $f: X \to Y$ is $\mathcal{F}$-continuous. This follows immediately from the lemma of §4 and from the observation of §2.

The same is true regardless of regularity of $Y$, provided (*) is replaced with the following stronger condition.

$(\star \star)$ For every set $A \subset X$, if each point $x \in A$ has a neighborhood $U(x)$ such that $U(x) \cap A \in \mathcal{F}$, then $A \in \mathcal{F}$.\(^2\)

More precisely, $(\star \star)$ implies that

(T) the family $\mathcal{T} = \{U \setminus A: U \text{ is open in } X, \ A \in \mathcal{F}\}$ is closed under arbitrary unions,\(^3\)

which, in turn, implies that every pointwise $\mathcal{F}$-continuous mapping defined on $X$ is $\mathcal{F}$-continuous.

Indeed, assume $(\star \star)$ and consider any union $E = \bigcup \{U_s: s \in S\}$ of members of $\mathcal{T}$. Put $U = \bigcup \{U_s: s \in S\}$ and $A = U \setminus E$. Since each $x \in A$ is in some $U_s$, and since $U_s \cap A \subset A_s \in \mathcal{F}$, the condition $(\star \star)$ yields $A \in \mathcal{F}$, and therefore $E = U \setminus A \in \mathcal{T}$. The other implication is equally straightforward.

6. An application. In 1923, Sierpiński and Zygmund [SZ] defined a function $h: \mathbb{R} \to \mathbb{R}$ whose restriction to every set of power $c$ is not continuous. The following is a more general setting of their result. (Compare [K2, p. 422].)

For every metric separable space $X$, and for every complete metric space $Y$ of power $c$, there exists a mapping $h: X \to Y$ whose restriction to every set of power $c$ is not continuous.

The theorem of this paper enables us to further improve that result as follows.

**Corollary.** Let $X$ be a metric separable space and let $Y$ be a complete metric space. Assume that $\mathcal{F}$ contains all subsets of $X$ having power $< c$. Then there exists a mapping $h: X \to Y$ such that for every set $X_0 \subset X$ with $X_0 \notin \mathcal{F}$, the restriction $f | X_0$ is not $\mathcal{F}_0$-continuous, $\mathcal{F}_0$ being the $\sigma$-ideal $\{A: A \in \mathcal{F}, A \subset X_0\}$.

\(^1\)Alternatively, $\mathcal{F}$ can be the $\sigma$-ideal of all meager sets in $X$.

\(^2\)This condition is still met when either $\mathcal{F}$ = meager sets (Banach Category Theorem), or $X$ is a hereditarily Lindelöf space.

\(^3\)And therefore $\mathcal{T}$ satisfies the axioms for a topology in $X$; $\mathcal{F}$-continuity then means continuity with respect to that richer topology in $X$. 

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PROOF. Let \( h \) be the mapping defined by Sierpiñski and Zygmund. Suppose \( h \mid X_0 \) is \( \mathcal{F}_0 \)-continuous for some \( X_0 \subset X \) with \( X_0 \not\in \mathcal{F} \). According to our theorem (case (II)), there is a set \( X_1 \subset X_0 \) such that \( X_0 \setminus X_1 \in \mathcal{F}_0 \) and \( h \mid X_1 \) is continuous. By the choice of \( h \), this set \( X_1 \) has power \( < c \). Therefore \( X_1 \in \mathcal{F} \), and consequently \( X_0 = (X_0 \setminus X_1) \cup X_1 \in \mathcal{F} \). This contradiction ends the proof.

A particular case can be obtained by letting \( \mathcal{F} \) contain all sets of power \( < c \) and no other sets.\(^4\) Then the property of the function \( h \) reads as follows.

Every set \( X_0 \) of power \( c \) contains a point \( x_0 \) such that \( h \) takes \( c \)-many points of every neighborhood of \( x_0 \) in \( X_0 \) outside some fixed neighborhood of \( f(x_0) \).

7. Remarks. A. In the case when \( \emptyset \) is the only member of \( \mathcal{F} \) that is open in \( X \), for instance if \( X \) is a Baire space and \( \mathcal{F} \) is the \( \sigma \)-ideal of all meager subsets of \( X \), condition (i) of §2 coincides with continuity of \( f \).

Also in that case, our lemma of §4 states that every (pointwise) \( \mathcal{F} \)-continuous mapping \( f : X \to Y \) is continuous, provided \( Y \) is regular.

B. One might be tempted to expect the following two statements to be equivalent as well, at least under some reasonable assumptions on \( X, Y \), and/or \( \mathcal{F} \).

(i)' \( f \mid X_0 \) is continuous for some open \( X_0 \subset X \) with \( X \setminus X_0 \in \mathcal{F} \),

(ii)' for every open \( V \subset Y \), \( f^{-1}(V) = U \cup A \) with \( U \) open in \( X \) and \( A \in \mathcal{F} \).

As in §2, (i)' implies (ii)' with no assumptions whatsoever. Unfortunately, the converse is false even when \( X = Y = \mathbb{R} \) (with natural topology) and \( \mathcal{F} \) is any of the standard \( \sigma \)-ideals in \( \mathbb{R} \). To see this, arrange all rational numbers in a sequence \( q_1, q_2, \ldots \) and put

\[
f(x) = \begin{cases} 
1/n & \text{if } x = q_n, \\
0 & \text{if } x \text{ is irrational.}
\end{cases}
\]

Let \( \mathcal{F} \) be any proper \( \sigma \)-ideal containing all rationals. The inverse image of every open set \( V \subset Y \) has a required form, since it is either open or a subset of the rationals. However, \( f \) is discontinuous on every open nonempty set.

REFERENCES


