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CONCERNING CONTINUITY APART FROM A MEAGER SET

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ABSTRACT. Given a σ-ideal $\mathscr{I}$ of subsets of a space X, mappings $f\colon X \to Y$ are investigated, such that $f \mid X_0$ is continuous for some closed $X_0 \subset X$ with $X \setminus X_0 \in \mathscr{I}$.

1. Introduction. Throughout the paper, X and Y are topological spaces, and $\mathscr{I}$ is a σ-ideal of subsets of X.

The following theorem was first proved by Kuratowski in 1930. (See **[K1]** and **[KM**, p. 408]; also **[EFK]** for a closely related result.)

If Y is second countable, then the statement

(a) *$f \mid X_0$ is continuous for some $X_0 \subset X$ with $X \setminus X_0 \in \mathscr{I}$ is equivalent to*

(b) *for every open $V \subset Y$, $f^{-1}(V) = (U \setminus A) \cup B$, where U is open in X, $A \in \mathscr{I}$, and $B \in \mathscr{I}$.*

We are interested in characterizing mappings that admit continuous restriction to a set $X_0 \subset X$ with an extra property in addition to $X \setminus X_0 \in \mathscr{I}$. The present paper is to investigate the case of a *closed* X_0.

2. The theorem. It is an easy observation that if

(i) *$f \mid X_0$ is continuous for some closed $X_0 \subset X$ with $X \setminus X_0 \in \mathscr{I}$,*

then

(ii) *for every open $V \subset Y$, $f^{-1}(V) = U \setminus A$ with U open in X and $A \in \mathscr{I}$.*

Indeed, let X_0 be as in (i) and take any closed $F \subset Y$. For (ii) to hold true, $f^{-1}(F)$ should have a form of $E \cup A$ with $E \subset X$ closed and $A \in \mathscr{I}$. Since

$$\begin{aligned} f^{-1}(F) &= \left[f^{-1}(F) \cap X_0\right] \cup \left[f^{-1}(F) \cap (X \setminus X_0)\right] \\ &= (f \mid X_0)^{-1}(F) \cup \left[f^{-1}(F) \cap (X \setminus X_0)\right], \end{aligned}$$

this is indeed the case.

We are now going to describe circumstances where (i) and (ii) are actually equivalent. To make some statements shorter, let us agree to say that f is *$\mathscr{I}$-continuous* whenever (ii) is satisfied.

THEOREM. *Let Y be a regular space and assume that either*

(I) *Y is second countable, or*

(II) *X is hereditarily Lindelöf, or*

(III) *$\mathscr{I}$ consists of all meager sets in X.*

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Let f: $X \to Y$ be $\mathscr{I}$-continuous. Then $f \mid X_0$ is continuous for some closed $X_0 \subset X$ with $X \setminus X_0 \in \mathscr{I}$.

3. Examples. Before a proof, let us look at two examples showing that the hypotheses of the theorem are to some extent indispensable; namely that

in (I), "second countable" cannot be weakened to "Lindelöf",

in (II), "hereditarily Lindelöf" cannot be weakened to "Lindelöf",

in (III), $\mathscr{I}$ cannot be just any σ-ideal, and regularity of Y is essential in all three versions.

A. Let $X = Y = \mathbf{R}$ with topologies as follows. All points except for 0 are isolated in both spaces, while neighborhoods of 0 in X (respectively, in Y) are complements of finite (respectively, countable) sets. These spaces X and Y are known to have nice topological properties: X is compact Hausdorff, Y is regular Hausdorff and Lindelöf.

Let $\mathscr{I}$ be the σ-ideal of all countable subsets of X. Although the identity function f: $X \to Y$ is obviously $\mathscr{I}$-continuous, its restriction to any closed, co-countable set $X_0 \subset X$ is discontinuous at 0.

B. Now consider $X = \mathbf{R}$ with its usual topology. Let $Y = \mathbf{R}$ have a richer topology consisting of all sets $U \setminus A$, where U is in the usual topology of $\mathbf{R}$ and A is a subset of $Z = \{1/n\colon n \in \mathbf{N}\}$. Thus Y is Hausdorff and second countable.

Let $\mathscr{I}$ be the σ-ideal of, say, all meager sets in X (or all sets of Lebesgue measure 0, or all countable sets). Since $Z \in \mathscr{I}$, the identity function f: $X \to Y$ is $\mathscr{I}$-continuous. On the other hand, the only closed set in X whose complement is a member of $\mathscr{I}$ is X itself. Therefore, the only restriction of f to be considered is f itself, and that is not continuous.

4. Proof of the theorem. In case (I), let $\{V_n\colon n \in \mathbf{N}\}$ be a base for the topology of Y. By the $\mathscr{I}$-continuity of f, to each V_n there correspond an open set $U_n \subset X$ and a set $A_n \in \mathscr{I}$ so that

$$f^{-1}(V_n) = U_n \setminus A_n. \tag{1}$$

Put

$$I = \bigcup \{A_n\colon n \in \mathbf{N}\} \tag{2}$$

and

$$X_0 = X \setminus \operatorname{int} I. \tag{3}$$

Of course, X_0 is closed in X and $X \setminus X_0 \subset I \in \mathscr{I}$. To show that $f \mid X_0$ is continuous, take any $x \in X_0$ and any neighborhood V of $f(x)$ in Y. Since Y is regular, there is an $n \in \mathbf{N}$ such that $f(x) \in V_n$ and

$$\operatorname{cl} V_n \subset V. \tag{4}$$

We will show that the corresponding neighborhood U_n of x in X, defined by (1), satisfies $f(U_n \cap X_0) \subset V$; this will prove that $f \mid X_0$ is continuous at x.

Suppose that, on the contrary, $f(x') \notin V$ for some $x' \in U_n \cap X_0$. Then, by (4), $f(x') \in Y \setminus \operatorname{cl} V_n$ and so, by the definition of a base, there is a V_m with $f(x') \in V_m$ and

$$V_m \cap V_n = \varnothing. \tag{5}$$

The corresponding set U_m contains x'; therefore

(6) $$x' \in U_m \cap U_n \cap X_0.$$

On the other hand, (1) and (2) yield $U_n \subset f^{-1}(V_n) \cup I$ and likewise, $U_m \subset f^{-1}(V_m) \cup I$. Hence, in view of (4),

$$U_n \cap U_m \subset f^{-1}(V_n \cap V_m) \cup I = I$$

and consequently, $U_n \cap U_m \subset \operatorname{int} I$. Now (3) gives $U_n \cap U_m \cap X_0 = \varnothing$, a contradiction to (6).

Cases (II) and (III) can be given a common proof. But first, let us recall a theorem known as Banach Category Theorem. (See **[B]** and **[O**, p. 62].)

If a subset $Z \subset X$ is locally meager then Z is meager.

An immediate corollary to the Banach Category Theorem is the following:

The union of all subsets of X that are open and meager at the same time is meager.

In other words, in case (III) of the theorem, the following condition (*) is satisfied:

(*) *The union of all members of I that are open in X is a member of I.*

The above holds in case (II) as well, since the union of all such sets (in a hereditarily Lindelöf space) coincides with the union of just countably many of them, which is in $\mathscr{I}$ by the definition of a σ-ideal. Thus (*) is a property that (II) and (III) have in common.

Let us say that f is *pointwise $\mathscr{I}$-continuous* if, for every $x \in X$ and for every neighborhood V of $f(x)$ in Y, there exists a neighborhood U of x in X such that $U \setminus f^{-1}(V) \in \mathscr{I}$.

Obviously, every $\mathscr{I}$-continuous mapping is pointwise $\mathscr{I}$-continuous. Therefore, to complete a proof of the theorem, it will suffice to prove the following lemma.

LEMMA. *Let Y be a regular space and let f be pointwise $\mathscr{I}$-continuous. Put*

$$X_0 = X \setminus \bigcup\{G \colon G \text{ is open in } X \text{ and } G \in \mathscr{I}\}.$$

Then $f \mid X_0$ is continuous.

PROOF. Consider any point $x \in X_0$ and any neighborhood V of $f(x)$ in Y. We need a neighborhood U of x in X such that $f(U \cap X_0) \subset V$.

First, using regularity of Y, we can select an open set W containing $f(x)$ with $\operatorname{cl} W \subset V$. Since f is pointwise $\mathscr{I}$-continuous, a required neighborhood U can now be chosen so that $U \setminus f^{-1}(W) \in \mathscr{I}$.

To show that $f(U \cap X_0) \subset V$, take any $x' \in U \cap X_0$; since $\operatorname{cl} W \subset V$, it will be enough to prove that $f(x') \in \operatorname{cl} W$.

Suppose that, on the contrary, there is a neighborhood W' of $f(x')$ such that $W \cap W' = \varnothing$. By the pointwise $\mathscr{I}$-continuity of f, we have $U' \setminus f^{-1}(W') \in \mathscr{I}$ for some neighborhood U' of x'. Hence

$$U \cap U' = (U \cap U') \setminus f^{-1}(W \cap W') \subset \left[U \setminus f^{-1}(W)\right] \cup \left[U' \setminus f^{-1}(W')\right] \in \mathscr{I},$$

and therefore $U \cap U' \subset X \setminus X_0$, by the definition of X_0.

We have reached a contradiction, because on the other hand, $x' \in U \cap U' \cap X_0$.

5. $\mathscr{I}$-continuity versus pointwise $\mathscr{I}$-continuity. As noted, every $\mathscr{I}$-continuous mapping is pointwise $\mathscr{I}$-continuous. The converse is not true in general.

For example, let $X = X_1 \times X_2$, where X_1 is a discrete uncountable space and $X_2 = \{0, 1, 1/2, 1/3, \dots\}$ with the usual topology. Let $\mathscr{I}$ be the σ-ideal of all countable subsets of X. Since the space X is locally countable, any mapping defined on it will automatically be pointwise $\mathscr{I}$-continuous. Not necessarily $\mathscr{I}$-continuous, though, as witnessed by the characteristic function of the set $X_1 \times \{0\}$. Notice that $(*)$ is not satisfied in this example.

For another example, let X and $\mathscr{I}$ be same as above,[1] except that X_2 is now **R** with the usual topology. Let Y be as in the second example of §3. It is easy to see that the mapping $f(x_1, x_2) = x_2$ is pointwise $\mathscr{I}$-continuous but not $\mathscr{I}$-continuous, although $(*)$ is now satisfied in X.

However, given both $(*)$ in X and regularity of Y, every pointwise $\mathscr{I}$-continuous mapping $f\colon X \to Y$ is $\mathscr{I}$-continuous. This follows immediately from the lemma of §4 and from the observation of §2.

The same is true regardless of regularity of Y, provided $(*)$ is replaced with the following stronger condition.

$(**)$ *For every set $A \subset X$, if each point $x \in A$ has a neighborhood $U(x)$ such that $U(x) \cap A \in \mathscr{I}$, then $A \in \mathscr{I}$.*[2]

More precisely, $(**)$ implies that

(T) *the family $\mathscr{T} = \{U \setminus A\colon\ U$ is open in $X,\ A \in \mathscr{I}\}$ is closed under arbitrary unions,*[3]

which, in turn, implies that every pointwise $\mathscr{I}$-continuous mapping defined on X is $\mathscr{I}$-continuous.

Indeed, assume $(**)$ and consider any union $E = \bigcup\{U_s \setminus A_s\colon\ s \in S\}$ of members of $\mathscr{T}$. Put $U = \bigcup\{U_s\colon\ s \in S\}$ and $A = U \setminus E$. Since each $x \in A$ is in some U_s and since $U_s \cap A \subset A_s \in \mathscr{I}$, the condition $(**)$ yields $A \in \mathscr{I}$, and therefore $E = U \setminus A \in \mathscr{T}$. The other implication is equally straightforward.

6. An application. In 1923, Sierpiński and Zygmund [**SZ**] defined a function $h\colon \mathbf{R} \to \mathbf{R}$ whose restriction to every set of power c is not continuous. The following is a more general setting of their result. (Compare [**K2**, p. 422].)

For every metric separable space X, and for every complete metric space Y of power c, there exists a mapping $h\colon\ X \to Y$ whose restriction to every set of power c is not continuous.

The theorem of this paper enables us to further improve that result as follows.

COROLLARY. *Let X be a metric separable space and let Y be a complete metric space. Assume that $\mathscr{I}$ contains all subsets of X having power $< c$. Then there exists a mapping $h\colon\ X \to Y$ such that for every set $X_0 \subset X$ with $X_0 \notin \mathscr{I}$, the restriction $f \mid X_0$ is not $\mathscr{I}_0$-continuous, $\mathscr{I}_0$ being the σ-ideal $\{A\colon\ A \in \mathscr{I}, A \subset X_0\}$.*

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[1]Alternatively, $\mathscr{I}$ can be the σ-ideal of all meager sets in X.

[2]This condition is still met when either $\mathscr{I} =$ meager sets (Banach Category Theorem), or X is a hereditarily Lindelöf space.

[3]And therefore $\mathscr{T}$ satisfies the axioms for a topology in X; $\mathscr{I}$-continuity then means continuity with respect to that richer topology in X.

PROOF. Let h be the mapping defined by Sierpiński and Zygmund. Suppose $h \mid X_0$ is $\mathscr{I}_0$-continuous for some $X_0 \subset X$ with $X_0 \notin \mathscr{I}$. According to our theorem (case (II)), there is a set $X_1 \subset X_0$ such that $X_0 \setminus X_1 \in \mathscr{I}_0$ and $h \mid X_1$ is continuous. By the choice of h, this set X_1 has power $< c$. Therefore $X_1 \in \mathscr{I}$, and consequently $X_0 = (X_0 \setminus X_1) \cup X_1 \in \mathscr{I}$. This contradiction ends the proof.

A particular case can be obtained by letting $\mathscr{I}$ contain all sets of power $< c$ and no other sets.[4] Then the property of the function h reads as follows.

Every set X_0 of power c contains a point x_0 such that h takes c-many points of every neighborhood of x_0 in X_0 outside some fixed neighborhood of $f(x_0)$.

7. Remarks. A. In the case when $\varnothing$ is the only member of $\mathscr{I}$ that is open in X, for instance if X is a Baire space and $\mathscr{I}$ is the σ-ideal of all meager subsets of X, condition (i) of §2 coincides with continuity of f.

Also in that case, our lemma of §4 states that every (pointwise) $\mathscr{I}$-continuous mapping $f\colon X \to Y$ is continuous, provided Y is regular.

B. One might be tempted to expect the following two statements to be equivalent as well, at least under some reasonable assumptions on X, Y, and/or $\mathscr{I}$.

(i)′ *$f \mid X_0$ is continuous for some open $X_0 \subset X$ with $X \setminus X_0 \in \mathscr{I}$,*

(ii)′ *for every open $V \subset Y$, $f^{-1}(V) = U \cup A$ with U open in X and $A \in \mathscr{I}$.*

As in §2, (i)′ implies (ii)′ with no assumptions whatsoever. Unfortunately, the converse is false even when $X = Y = \mathbf{R}$ (with natural topology) and $\mathscr{I}$ is any of the standard σ-ideals in $\mathbf{R}$. To see this, arrange all rational numbers in a sequence $q_1, q_2, \ldots$ and put

$$f(x) = \begin{cases} 1/n & \text{if } x = q_n, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Let $\mathscr{I}$ be any proper σ-ideal containing all rationals. The inverse image of every open set $V \subset Y$ has a required form, since it is either open or a subset of the rationals. However, f is discontinuous on every open nonempty set.

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[4] This $\mathscr{I}$ is a σ-ideal by König's theorem.