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PROPERTY C, REFINABLE MAPS AND DIMENSION RAISING MAPS

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ABSTRACT. We show that refinable maps defined on compacta preserve Property C. H. Kato has proved the analogous result for weakly infinite dimensional spaces. We also show that if f is a map from a compact C space X onto a non C space Y, then the set of points in Y with an uncountable number of preimages is a space that does not have Property C.

1. Introduction. Property C is a covering property introduced by Haver [**H1, H2**] and further investigated by Addis and Gresham [**AG, Gr**]. Every countable dimensional space is a C space and each infinite dimensional C space is weakly infinite dimensional. It is unknown whether Property C is equivalent to weak infinite dimensionality. However, R. Pol's example [**Po**] shows that Property C is not equivalent to countable dimensionality.

This example and results of F. D. Ancel [**A1, A2**] have focused renewed interest on C spaces. Ancel shows that any cell-like dimension raising map defined on an ANR must have a non C space as its image. A better understanding of the relationship between Property C and weak infinite dimensionality should lead to a better understanding of potential images of cell-like dimension raising maps.

Refinable maps have been investigated in [**FR, FK, K1, K2, Pa, Ko1** and **Ko2**]. H. Kato shows that refinable maps defined on compacta preserve weak infinite dimensionality [**K3**]. A. Koyama in [**Ko2**] asks whether refinable maps on compacta preserve Property C and adds in proof a statement that they do.

The main result presented here, Theorem 1, shows that refinable maps on compacta preserve Property C. K. Sakai in a personal communication has shown that refinable maps are approximately invertible. This result combined with Ancel's work in [**A2**] then provides another proof of Theorem 1.

In [**Ga**], it is shown that a map f from a σ compact C space X onto a non C space Y has the property that Y^* is infinite dimensional. Here Y^* is $\{y \in Y \mid f^{-1}(y)$ has cardinality $\geqslant c\}$. This parallels results for weakly infinite dimensional spaces presented in [**SK** and **Va**]. In [**Le**], Leibo proves the following result. If f is a map from a compact weakly infinite dimensional space X onto a strongly infinite

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dimensional space Y, then Y^* is strongly infinite dimensional. In Theorem 2, we show that the analogous result holds for spaces with Property C.

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2. Definitions. All spaces will be separable metric. A space X has *Property* C or is a *C space* if for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open covers of X, there exists a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$ satisfying the following conditions:

1. Each $\mathcal{V}_i$ is a pairwise disjoint collection of open sets.
2. Each V in $\mathcal{V}_i$ is contained in some U in $\mathcal{U}_i$.
3. $\bigcup \mathcal{V}_i$ is a cover of X.

Such a sequence is called a *C refinement* of the sequence of covers $(\mathcal{U}_i)$.

A *map* between spaces is a continuous function. An *ANR* is an absolute neighborhood retract for the class of metric spaces. A map $f\colon X \to Y$ between compacta is an ε map for some $\varepsilon > 0$ if, for each $y \in Y$, $\operatorname{diam}(f^{-1}(y)) < \varepsilon$. A map $r\colon X \to Y$ between compacta is *refinable* if for each $\varepsilon > 0$ there exists a surjective ε map, called an *ε refinement* of r, $r_\varepsilon\colon X \to Y$, so that $d(r, r_\varepsilon) < \varepsilon$.

If $\{A_i\}$, $1 \leqslant i < \infty$, is a sequence of closed sets in a compactum X, the $\limsup_i A_i$ is $\{x \in X \mid \text{for each open set } U \text{ containing } x, U \text{ intersects infinitely many of the } A_i\}$. Equivalently, $\limsup_i A_i = \{x \in X \mid \text{there exists } x_{i_k} \in A_{i_k} \text{ with } \lim_{k \to \infty} x_{i_k} = x\}$. Note that if U is an open set in a compactum X so that $\limsup_i A_i \subset U$, then there exists an N so that $A_n \subset U$ for each $n > N$.

If $f\colon X \to Y$ is a map, Y^* is defined to be $\{y \in Y \mid f^{-1}(y) \text{ has cardinality} \geqslant c\}$. Here c is the cardinality of $\mathbf{R}$.

3. Refinable maps. In proving Theorem 1, we need the following result due to Loncar and Mardesic.

LEMMA 1 [**LM**]. *Let $f\colon X \to A$ be a map from a compactum X to an ANR A. Given any $\varepsilon > 0$, there exists a $\delta > 0$ so that for any surjective δ map $g\colon X \to Y$ there exists a map $h\colon Y \to A$ with $d(f, hg) < \varepsilon$.*

THEOREM 1. *Let $r\colon X \twoheadrightarrow Y$ be a surjective refinable map between compacta. If X has Property* C, *then Y has Property* C.

PROOF. For each $i \in \mathbf{N}$ let $\mathcal{U}_i$ be an open cover of Y. Since Y is compact, we may assume each $\mathcal{U}_i = \{U_{ij}\}$, $1 \leqslant j \leqslant N_i$. For each $i \in \mathbf{N}$ let $r^{-1}(\mathcal{U}_i) = \{r^{-1}(U) \mid U \in \mathcal{U}_i\}$. Since X is compact and has Property C, there exists an integer N and collections of open sets $\mathcal{V}_i$ of X, $1 \leqslant i \leqslant N$, so that

1. For each i, $1 \leqslant i \leqslant N$, $\mathcal{V}_i = \{\mathcal{V}_{ij}\}$, $1 \leqslant j \leqslant N_i$, is a collection of open sets with pairwise disjoint closures.
2. For each i, $1 \leqslant i \leqslant N$, and for each j, $1 \leqslant j \leqslant N_i$, $V_{ij} \subset \overline{V}_{ij} \subset r^{-1}(U_{ij})$.
3. $\bigcup_{i=1}^N \mathcal{V}_i$ is an open cover of X.

Now for each $i = 1, \ldots, N$ let A_i denote the join of N_i intervals, joined at a common endpoint. Label the remaining endpoints $a_1, a_2, \ldots, a_{N_i}$. To avoid ambiguity, an endpoint a_j will always be referred to together with the A_i of which it is an endpoint. The A_i are of course ANRs.

For each $i = 1, \ldots, N$, we can choose maps f_i: $X \to A_i$ so that, for each $j = 1, \ldots, N_i$,

$$f_i^{-1}(a_j) = r^{-1}\Big(Y \setminus \bigcup_{k \neq j} U_{ik}\Big) \cup \overline{V}_{ij}.$$

Since $f_i^{-1}(a_j) \subset r^{-1}(U_{ij})$ we see that $f_i(r^{-1}(Y \setminus U_{ij}))$ is a closed set in A; disjoint from the endpoint a_j.

Using complete normality, for each i, $1 \leqslant i \leqslant N$, we can choose open sets M_{ij} and N_{ij}, $1 \leqslant j \leqslant N_i$, in A_i so that

1. $a_j \in N_{ij}$.
2. For each i, $\{N_{ij} \mid i \leqslant j \leqslant N_i\}$ is a pairwise disjoint collection.
3. $f_i(r^{-1}(Y \setminus U_{ij})) \subset M_{ij}$.
4. $d(\overline{M_{ij}}, \overline{N_{ij}}) \equiv D_{ij} > 0$.

Fix $k \in \mathbf{N}$ and $i \in \{1, \ldots, N\}$. For each such i and k, the lemma guarantees the existence of a number $\delta_{ik} > 0$ so that if g: $X \to Y$ is any surjective δ_{ik} map then there exists a map h_{ik}: $Y \to A_i$ with $d(f_i, h_{ik} \circ g) < 1/k$. In particular, if we let $\delta_k = \min\{1/k, \delta_k\}$, $1 \leqslant i \leqslant N$, then the refinability of r allows us to choose a map r_k: $X \to Y$ so that r_k is a δ_k refinement of r. In addition, since r_k is then a δ_{ik} map for each i, $1 \leqslant i \leqslant N$, Lemma 1 can be applied to choose maps h_{ik}: $Y \to A_i$ with $d(f_i, h_{ik} \circ r_k) < 1/k$.

We thus obtain two sequences of maps. We obtain a sequence (r_k) of maps from X onto Y where r_k is a $1/k$ refinement of r. We also obtain for each i, $1 \leqslant i \leqslant N$, a sequence (h_{ik}) of maps from Y to A_i so that $d(f_i, h_{ik} \circ r_k) < 1/k$.

CLAIM. $\limsup_k(r_k^{-1}(Y \setminus U_{ij})) \subset r^{-1}(Y \setminus U_{ij}) \subset f_i^{-1}(M_{ij})$.

For, let $x \in \limsup_k r_k^{-1}(Y \setminus U_{ij})$. Then there exists a sequence $x_{k_l} \in r_{k_l}(Y \setminus U_{ij})$ so that $x_{k_l} \to x$ as $l \to \infty$. Now,

$$d\big(r(x), Y \setminus U_{ij}\big) \leqslant d\big(r_{k_l}(x_{k_l}), Y \setminus U_{ij}\big) + d\big(r_{k_l}(x_{k_l}), r(x_{k_l})\big) + d\big(r(x_{k_l}), r(x)\big).$$

This goes to 0 as $l \to \infty$. Since $Y \setminus U_{ij}$ is closed, this completes the proof of the claim.

It is now possible to choose m large enough so that for each i, $1 \leqslant i \leqslant N$, and for each j, $1 \leqslant j \leqslant N_i$,

$$m^{-1} < d\big(a_j, A_i \setminus N_{ij}\big), \quad m^{-1} < D_{ij} \quad \text{and} \quad r_m^{-1}\big(Y \setminus U_{ij}\big) \subset f_i^{-1}\big(M_{ij}\big).$$

Finally, let $W_{ij} = h_{im}^{-1}(N_{ij})$ and $\mathscr{W}_i = \{W_{ij} \mid 1 \leqslant j \leqslant N_i\}$.

For any fixed i, $\{N_{ij} \mid 1 \leqslant j \leqslant N_i\}$ is a collection of pairwise disjoint open sets in A_i. Thus each $\mathscr{W}_i$ is also a collection of pairwise disjoint open sets in Y. To show that Y has Property C, it only remains to show that $\mathscr{W}_i$ refines $\mathscr{U}_i$ for each i, and that $\bigcup_{i=1}^N \mathscr{W}_i$ covers Y.

Suppose $y \in W_{ij}$ and that $y \in Y \setminus U_{ij}$. Let $x \in r_m^{-1}(y)$. By our choice of m, $x \in f_i^{-1}(M_{ij})$, so that $f_i(x) \in \overline{M_{ij}}$. Also, $h_{im}(y) = h_{im} \circ r_m(x) \in N_{ij}$, but then $d(f_i(x), h_{im} \circ r_m(x)) \geqslant d(\overline{M_{ij}}, \overline{N_{ij}}) = D_{ij} > 1/m$, which is a contradiction. Thus $\mathscr{W}_i$ refines $\mathscr{U}_i$.

Next, let $y \in Y$ and $x \in r_m^{-1}(y)$. Then there exists i and j so that $x \in V_{ij}$. Now

$$d(f_i(x), h_{im} \circ r_m(x)) = d(a_j, h_{im}(y)) < 1/m$$
$$< d(a_j, A_i \setminus N_{ij}).$$

Since $a_j \leqslant N_{ij} \subset A_i$, we have $h_{im}(y) \leqslant N_{ij}$. Thus $y \leqslant h_{im}^{-1}(N_{ij}) = W_{ij}$ and $\bigcup_{i=1}^{N} \mathscr{W}_i$ covers Y. So Y has Property C. □

In **[Ro]**, D. Rohm shows that weak infinite dimensonality can be defined using a covering property similar to that used in defining Property C. The difference is that binary open covers rather than arbitrary open covers are used. Using this definition, the above proof provides an alternate proof of Kato's Theorem that refinable maps preserve weak infinite dimensionality **[K3]**.

4. Dimension raising maps. If Property C is interpreted as a dimension like property, Theorem 1 can be read to say that refinable maps on C spaces do not raise dimension. Theorem 2 explains properties of closed dimension raising maps defined on C spaces.

LEMMA 2. *If X is a non C space and A is a subspace of X that has Property* C, *then there exists a subspace F of $X \setminus A$ that is a non C space and that is closed in X.*

PROOF. Let $\mathscr{U}_i$, $1 \leqslant i < \infty$, be a sequence of open covers of X that has no C refinement. Then the sequence of covers $\mathscr{U}_{2i}$, $1 \leqslant i < \infty$, restricted to A does have a C refinement $\mathscr{V}_{2i}$, $1 \leqslant i < \infty$. We may assume that the sets V in $\mathscr{V}_{2i}$ are actually pairwise disjoint and open in X. Let $V = \bigcup_{i=1}^{\infty}(\bigcup \mathscr{V}_{2i})$. Then $\mathscr{V}$ is an open subset of X that contains A. Let $F = X \setminus \mathscr{V}$. Then the sequence of covers $\mathscr{U}_{2i+1}$, $0 \leqslant i < \infty$, restricted to F cannot have a C refinement. Otherwise, the sequence of covers $\mathscr{U}$, $1 \leqslant i < \infty$, of X would have a C refinement. So F is a non C space. □

THEOREM 2. *Let f be a closed map from a σ compact C space X onto a non C space Y. Then Y^* does not have Property* C.

PROOF. By **[Ga]** Y^* is nonempty. Assume that Y^* had Property C. The lemma then implies that there is a non C subspace F of $Y \setminus Y^*$ so that F is closed in Y. Let $Z = f^{-1}(F)$. Then $f \mid Z\colon Z \to F$ is a map satisfying the hypotheses of the theorem. So by **[Ga]**, there exists a $y \in F$ so that $f^{-1}(y)$ has cardinality $\geqslant c$. This contradicts the fact that $y \in F \subset Y \setminus Y^*$. □

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