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# A WEAKLY INFINITE-DIMENSIONAL SPACE WHOSE PRODUCT WITH THE IRRATIONALS IS STRONGLY INFINITE-DIMENSIONAL

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ABSTRACT. We give an example of a weakly infinite-dimensional space $X$ such that the product $X \times B$ of $X$ and a subspace $B$ of the irrationals is strongly infinite-dimensional; under the assumption of the Continuum Hypothesis, $B$ can be the irrationals. This example answers a question of Addis and Gresham [**AG**].

**1. Terminology and notation.** All spaces under discussion are metrizable and separable. Our terminology follows [**AP** and **E**]. We denote by $I$ the real interval $[0,1]$, by $C$ the usual Cantor set in $I$ and by $I^\omega$ the Hilbert cube. We denote the space of the irrational numbers from $I$ by $P$ and the rational numbers from $I$ by $Q$.

A space $X$ is weakly infinite-dimensional [**AP**, Chapter 10, §§4–7] if for every sequence $\{(A_1, B_1), (A_2, B_2), \dots\}$ of pairs of closed disjoint subsets of $X$ there are partitions $L_i$ in $X$ between $A_i$ and $B_i$ such that $\bigcap_{i=1}^\infty L_i = \varnothing$. Otherwise, $X$ is strongly infinite-dimensional.

A space $X$ is called a $C$-space if for every sequence $\mathscr{G}_1, \mathscr{G}_2, \dots$ of open covers of $X$ there exists a sequence $\mathscr{U}_1, \mathscr{U}_2, \dots$ of families of open subsets of $X$ such that, for $i = 1, 2, \dots,$

(i) the members of $\mathscr{U}_i$ are pairwise disjoint,

(ii) each member of $\mathscr{U}_i$ is contained in a member of $\mathscr{G}_i$,

(ii) the union $\bigcup_{i=1}^\infty \mathscr{U}_i$ covers $X$.

The notion of $C$-space was introduced by W. Haver in [**H**] for metric space and by D. Addis and J. Gresham in [**AG**] for general topological spaces.

LEMMA 1 [**AG**]. *Every C-space is weakly infinite-dimensional.*

**2. Results.** The aim of this note is to construct the following examples.

EXAMPLE 1. *There exists a weakly infinite-dimensional space $X$ such that the product $X \times B$ of $X$ with a certain subspace $B$ of the space of irrationals is strongly infinite-dimensional. Moreover, $X$ is a C-space while $X \times B$ is not a C-space.*

EXAMPLE 2. *Under the assumption of the Continuum Hypothesis there exists a weakly infinite-dimensional space $X$ such that the product $X \times P$ of $X$ with the space of irrationals $P$ is strongly infinite-dimensional. Moreover, $X$ is a C-space while $X \times P$ is not a C-space.*

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Example 2 gives, in particular, an answer to a question formulated by Addis and Gresham in [**AG**] as to whether the product of a $C$-space and the space of irrationals is a $C$-space.

The first example of two metrizable separable weakly infinite-dimensional spaces $X_1$ and $X_2$ (being also $C$-spaces) whose product $X_1 \times X_2$ is strongly infinite-dimensional was given by R. Pol in [**P2**] (see [**EP**, Example 8.20] for the proof that $X_i$ are $C$-spaces).

**3. Constructions.** Our constructions are based on the following lemma, which follows an idea of Michael [**M**].

LEMMA 2. *Let $f\colon X \to Y$ be a mapping of a space $X$ into a space $Y$ and let $A = f^{-1}(B)$ for some $B \subset Y$. If the subset* $\operatorname{Graph}(f \mid A) = \{(x, f(x))\colon x \in A\}$ *of $X \times Y$ is strongly infinite-dimensional, then the product $X \times B$ is strongly infinite-dimensional* (*and thus is not a $C$-space*).

The proof of this lemma follows from the fact that a closed subspace of a weakly infinite-dimensional space is weakly infinite-dimensional and that $X \times B$ contains $\operatorname{Graph}(f \mid A)$ as a closed subset.

*Construction of Example* 1. Let $T$ be the weakly infinite-dimensional compactum, which is not countable dimensional defined by R. Pol in [**P1**]. The space $T$ has the following structure: $T = Y \cup Z$, where $Y$ is a completely metrizable totally disconnected space, which is strongly infinite-dimensional, and $Z$ is the union of countably many 0-dimensional sets $Z_1, Z_2, \ldots$. Moreover, we can assume (see [**P1**, Comment B]) that $T$ is embedded in the product $C \times I^\omega$ of the Cantor set $C$ and the Hilbert cube $I^\omega$, in such a way that $Y$ is the graph of a certain first-Baire-class function $f\colon C \to I^\omega$, i.e., $Y = \{(t, f(t))\colon t \in C\} \subset C \times I^\omega$.

Split the Cantor set $C$ into two disjoint Bernstein sets $B_1$ and $B_2$ (see [**K**, §40]), i.e., all compact subsets of $B_i$ are countable. Since $Y$ is strongly infinite-dimensional, either $Y_1 = \{(t, f(t))\colon t \in B_1\}$ or $Y_2 = \{(t, f(t))\colon t \in B_2\}$ is strongly infinite-dimensional. Suppose that $Y_2$ is strongly infinite-dimensional and let $X$ be the following subspace of $C \times C \times I^\omega$: $X = (B_1 \times T) \cup \{(t, t, f(t))\colon t \in B_2\} \subset C \times T$. Denote $X_1 = (B_1 \times T)$, $X_2 = \{(t, t, f(t))\colon t \in B_2\}$. The set $X_2$ is obviously homeomorphic to $Y_2$, hence it is strongly infinite dimensional. We will show that

$$X \text{ is a } C\text{-space.} \tag{1}$$

Suppose that $\mathscr{G}_1, \mathscr{G}_2, \ldots$ is a sequence of open covers of $X$. Fix an $i \geqslant 3$. Since the set $B_1 \times Z_{i-2}$ is 0-dimensional, there exists a family $\mathscr{W}_i$ of pairwise disjoint sets open in $C \times T$ such that $B_1 \times Z_{i-2} \subset \bigcup \mathscr{W}_i = W_i$ and for each $W \in \mathscr{W}_i$ the set $W \cap X$ is contained in some member of $\mathscr{G}_i$. Let $F = (B_1 \times T) \setminus \bigcup_{i=3}^\infty W_i$; we will show that $\dim F = 0$. Since $\bigcup_{i=3}^\infty W_i \supset B_1 \times Z$, $F$ is a closed subset of $C \times T$ contained in $B_1 \times Y$. Consider the projection $p_1\colon B_1 \times T \to B_1$. Since $T$ is compact, $p_1$, as well as $p = p_1/F\colon F \to B_1$, is a closed mapping. Moreover, for each $t \in B_1$ the set $p^{-1}(t) = (\{t\} \times T) \cap F$ is a compact set contained in $\{t\} \times Y$. It follows that $\dim p^{-1}(t) = 0$, because $Y$ is totally disconnected. Thus, by the Hure-

wicz Theorem (see [E, Theorem 1.12.4])

$$\dim F \leqslant \sup\{\dim p^{-1}(t)\colon t \in B_1\} + \dim B_1 = 0.$$

Now, let $\mathscr{W}_2$ be a family of pairwise disjoint open subsets of $C \times T$ such that $F \subset \bigcup \mathscr{W}_2 = W_2$ and for each $W \in \mathscr{W}_2$ there exists $U \in \mathscr{G}_2$ such that $W \cap X \subset U$. Put $K = X \setminus \bigcup_{i=2}^{\infty} W_i$; since $\bigcup_{i=2}^{\infty} W_i \supset B_1 \times T$, then $K \subset X_2$. We shall show that the set $K$ is countable. Let $p_2\colon C \times T \to C$ be the projection. Since $p_2$ is a closed mapping, the set $L = p_2((C \times T) \setminus \bigcup_{i=2}^{\infty} W_i)$ is closed in $C$. Moreover, the set $L$ is contained in $B_2$, hence it is countable. It follows that the set $p_2(K) \subset L$ is countable, which implies that $K$ is countable, because $p_2 | X_2\colon X_2 \to B_2$ is a one-to-one mapping. Let $\mathscr{U}_1$ be a family of pairwise disjoint open subsets of $X$ such that $K \subset \bigcup \mathscr{U}_1$ and each member of $\mathscr{U}_1$ is contained in some member of $\mathscr{G}_1$. Then the family $\mathscr{U}_1, \mathscr{U}_2, \ldots,$ where $\mathscr{U}_i = \{W \cap X\colon W \in \mathscr{W}_i\}$ for $i \geqslant 2$, satisfies conditions (i), (ii) and (iii).

Finally, let $f\colon X \to C$ be the projection. Then $f^{-1}(B_2) = X_2$ and the Graph$(f | X_2)$ is homeomorphic to $X_2$ in a natural way, hence it is strongly infinite-dimensional. It follows by Lemma 2 that the product $X \times B_2$ is strongly infinite-dimensional. Since $B_2$ is a 0-dimensional metrizable separable space, it is homeomorphic to a subset $B$ of the irrationals $P$.

*Construction of Example* 2. In this example we use, among others, some ideas of **[P2]**.

Let $Z = \prod_{i=0}^{\infty} I_i$, where $I_i = I$ for $i = 0, 1, 2, \ldots,$ be the Hilbert cube and the mappings $p_0\colon Z \to I_0$ and $p_n\colon \prod_{i=1}^{\infty} I_i \to \prod_{i=1}^{n} I_i$ be appropriate projections. Arrange all rational numbers in $I_0$ into a sequence $q_1, q_2, \ldots$. Denote by $T$ a compactum obtained from $Z$ by attaching to each compactum $\{q_n\} \times \prod_{i=1}^{\infty} I_i$ the $n$-cube $\prod_{i=1}^{n} I_i$ by the map $p_n$, i.e., $T$ is a quotient space defined by an upper semicontinuous decomposition consisting of singletons $\{(t, x)\}$, where $t \in P$ and $x \in \prod_{i=1}^{\infty} I_i$, and the sets $\{q_n\} \times p_n^{-1}(y)$, where $y \in \prod_{i=1}^{n} I_i$. Let $\pi\colon Z \to T$ be the natural quotient mapping and $X_1 = \pi(Q \times \prod_{i=1}^{\infty} I_i)$.

Using the Continuum Hypothesis, define a transfinite sequence $G_1 \supset G_2 \supset \cdots \supset G_\alpha \supset \cdots$, $\alpha < \omega_1$, of $G_\delta$-subsets of $T$ containing $X_1$ and such that for every $G_\delta$-set $G$ in $T$ containing $X_1$ there exists $\alpha$ with $G_\alpha \subset G$ ($\omega_1$ denotes the first uncountable ordinal). Again by CH, we can arrange all continua in $Z$ intersecting $\{0\} \times \prod_{i=1}^{\infty} I_i$ and $\{1\} \times \prod_{i=1}^{\infty} I_i$ into a sequence $K_1, K_2, \ldots, K_\alpha, \ldots, \alpha < \omega_1$.

By transfinite induction we will choose, for each $\alpha < \omega_1$, a point $y_\alpha \in K_\alpha \cap \pi^{-1}(G_\alpha) \cap p_0^{-1}(P)$ such that $y_\alpha \notin \{y_\beta\colon \beta < \alpha\}$. Suppose that $\alpha = 0$ or that we have already defined all points $y_\beta$ for $\beta < \alpha$. Since $\pi^{-1}(G_\alpha)$ is a $G_\delta$-set in $Z$ containing $Q \times \prod_{i=1}^{\infty} I_i$, $M = p_0(Z \setminus \pi^{-1}(G_\alpha))$ is an $F_\sigma$-set in $I_0$, contained in $P$ (notice that the projection $p_0$ is closed). Thus the set $P \setminus M$ is of second category in $I_0$. Hence there exists $p_\alpha \in (P \setminus M) \setminus \{p_0(y_\beta)\colon \beta < \alpha\}$. Since $\{p_\alpha\} \times \prod_{i=1}^{\infty} I_i \subset \pi^{-1}(G_\alpha)$ and the continuum $K_\alpha$ intersects $\{p_\alpha\} \times \prod_{i=1}^{\infty} I_i$, there exists $y_\alpha \in K_\alpha \cap \pi^{-1}(G_\alpha) \cap p_0^{-1}(p_\alpha)$. If we put $x_\alpha = \pi(y_\alpha)$ for every $\alpha < \omega_1$, then

$$x_\alpha \in G_\alpha \cap \pi(K_\alpha) \setminus \big(X_1 \cup \{x_\beta\colon \beta < \alpha\}\big).$$

*Claim*. The subspace $X = X_1 \cup \{x_\alpha\colon \alpha < \omega_1\}$ of $T$ has the desired properties. We will show that

(2) $X$ is a $C$-space,

and

(3) the subspace $X_2 = \{x_\alpha\colon \alpha < \omega_1\}$ of $X$ is strongly infinite dimensional.

To show (2) suppose that $\mathscr{G}_1, \mathscr{G}_2, \ldots$ is a sequence of open covers of $X$. Decompose the set $X_1$ into a sequence $Z_1, Z_2, \ldots$ of 0-dimensional sets. For $i = 2, 3, \ldots$ choose a family $\mathscr{W}_i$ of disjoint open subsets of $T$ such that $Z_{i-1} \subset W_i = \bigcup \mathscr{W}_i$ and for each $U \in \mathscr{W}_i$ there exists $G \in \mathscr{G}_i$ satisfying $U \cap X \subset G$ (see [**E**, Lemma 1.7.3]). Then $X_1 \subset \bigcup_{i=2}^\infty W_i$ and there exists $\alpha$ such that $G_\alpha \subset \bigcup_{i=2}^\infty W_i$. The set $X \setminus \bigcup_{i=2}^\infty W_i \subset X \setminus G_\alpha$ is contained in $\{x_\beta\colon \beta < \alpha\}$, hence it is countable. Thus there exists a family $\mathscr{U}_1$ of open subsets of $X$ satisfying conditions (i) and (ii) for $i = 1$ and such that $U_1 = \bigcup \mathscr{U}_1 \supset X \setminus \bigcup_{i=2}^\infty W_i$. The families $\mathscr{U}_1, \mathscr{U}_2, \ldots,$ where $\mathscr{U}_i = \{W \cap X\colon W \in \mathscr{W}_i\}$ for $i \geqslant 2$, satisfy conditions (i), (ii), and (iii).

To prove (3) we will show that the subspace $\pi^{-1}(X_2)$ of the Hilbert cube, which is homeomorphic to $X_2$, is strongly infinite dimensional. For $n = 1, 2, \ldots,$ let $A_n = \{\{x_i\} \in \prod_{i=0}^\infty I_i\colon x_n = 0\}$ and $B_n = \{\{x_i\} \in \prod_{i=0}^\infty I_i\colon x_n = 1\}$ be opposite faces of the Hilbert cube $Z$ and let $U_i$ and $V_i$ be open subsets of $Z$ such that $A_i \subset U_i$, $B_i \subset V_i$, and $\overline{U}_i \cap \overline{V}_i = \varnothing$. For $i = 1, 2, \ldots,$ let $L_i$ be an arbitrary partition between $\overline{U}_i \cap \pi^{-}(X_2)$ and $\overline{V}_i \cap \pi^{-1}(X_2)$ in $\pi^{-1}(X_2)$. The partition $L_i$ can be extended (see [**E**, Lemma 1.2.9]) to a partition $L_i'$ between $A_i$ and $B_i$ in $Z$ satisfying $L_i' \cap \pi^{-1}(X_2) \subset L_i$. Since $\bigcap_{i=1}^\infty L_i'$ contains a continuum connecting $\{0\} \times \prod_{i=1}^\infty I_i$ and $\{1\} \times \prod_{i=1}^\infty I_i$ (see [**RSW**, Lemma 5.2]), say $K_\alpha$, we have $y_\alpha \in \bigcap_{i=1}^\infty L_i' \cap \pi^{-1}(X_2)$. Therefore $\bigcap_{i=1}^\infty L_i \neq \varnothing$. This ends the proof of (3).

Finally, let $f\colon X \to I_0$ be the projection. Then $f^{-1}(P) = X_2$. Moreover, $\mathrm{Graph}(f \mid X_2)$ is homeomorphic to $X_2$, hence it is strongly infinite-dimensional. Thus, by Lemma 2, the product $X \times P$ is strongly infinite-dimensional.

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