A WEAKLY INFINITE-DIMENSIONAL SPACE WHOSE PRODUCT WITH THE IRRATIONALS IS STRONGLY INFINITE-DIMENSIONAL

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Abstract. We give an example of a weakly infinite-dimensional space $X$ such that the product $X \times B$ of $X$ and a subspace $B$ of the irrationals is strongly infinite-dimensional; under the assumption of the Continuum Hypothesis, $B$ can be the irrationals. This example answers a question of Addis and Gresham [AG].

1. Terminology and notation. All spaces under discussion are metrizable and separable. Our terminology follows [AP and E]. We denote by $I$ the real interval $[0,1]$, by $C$ the usual Cantor set in $I$ and by $I^\omega$ the Hilbert cube. We denote the space of the irrational numbers from $I$ by $P$ and the rational numbers from $I$ by $Q$.

A space $X$ is weakly infinite-dimensional [AP, Chapter 10, §§4–7] if for every sequence $\{(A_1, B_1), (A_2, B_2), \ldots \}$ of pairs of closed disjoint subsets of $X$ there are partitions $L_i$ of $X$ between $A_i$ and $B_i$ such that $\cap_{i=1}^\infty L_i = \emptyset$. Otherwise, $X$ is strongly infinite-dimensional.

A space $X$ is called a C-space if for every sequence $\mathcal{G}_1, \mathcal{G}_2, \ldots$ of open covers of $X$ there exists a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of families of open subsets of $X$ such that, for $i = 1, 2, \ldots$,

(i) the members of $\mathcal{U}_i$ are pairwise disjoint,
(ii) each member of $\mathcal{U}_i$ is contained in a member of $\mathcal{G}_i$,
(iii) the union $\cup_{i=1}^\infty \mathcal{U}_i$ covers $X$.

The notion of C-space was introduced by W. Haver in [H] for metric space and by D. Addis and J. Gresham in [AG] for general topological spaces.

Lemma 1 [AG]. Every C-space is weakly infinite-dimensional.

2. Results. The aim of this note is to construct the following examples.

Example 1. There exists a weakly infinite-dimensional space $X$ such that the product $X \times B$ of $X$ with a certain subspace $B$ of the space of irrationals is strongly infinite-dimensional. Moreover, $X$ is a C-space while $X \times B$ is not a C-space.

Example 2. Under the assumption of the Continuum Hypothesis there exists a weakly infinite-dimensional space $X$ such that the product $X \times P$ of $X$ with the space of irrationals $P$ is strongly infinite-dimensional. Moreover, $X$ is a C-space while $X \times P$ is not a C-space.
Example 2 gives, in particular, an answer to a question formulated by Addis and Gresham in [AG] as to whether the product of a C-space and the space of irrationals is a C-space.

The first example of two metrizable separable weakly infinite-dimensional spaces $X_1$ and $X_2$ (being also C-spaces) whose product $X_1 \times X_2$ is strongly infinite-dimensional was given by R. Pol in [P2] (see [EP, Example 8.20] for the proof that $X_i$ are C-spaces).

3. Constructions. Our constructions are based on the following lemma, which follows an idea of Michael [M].

**Lemma 2.** Let $f: X \to Y$ be a mapping of a space $X$ into a space $Y$ and let $A = f^{-1}(B)$ for some $B \subset Y$. If the subset $\text{Graph}(f|A) = \{(x,f(x)): x \in A\}$ of $X \times Y$ is strongly infinite-dimensional, then the product $X \times B$ is strongly infinite-dimensional (and thus is not a C-space).

The proof of this lemma follows from the fact that a closed subspace of a weakly infinite-dimensional space is weakly infinite-dimensional and that $X \times B$ contains $\text{Graph}(f|A)$ as a closed subset.

**Construction of Example 1.** Let $T$ be the weakly infinite-dimensional compactum, which is not countable dimensional defined by R. Pol in [P1]. The space $T$ has the following structure: $T = Y \cup Z$, where $Y$ is a completely metrizable totally disconnected space, which is strongly infinite-dimensional, and $Z$ is the union of countably many 0-dimensional sets $Z_1, Z_2, \ldots$. Moreover, we can assume (see [P1, Comment B]) that $T$ is embedded in the product $C \times I^\omega$ of the Cantor set $C$ and the Hilbert cube $I^\omega$, in such a way that $Y$ is the graph of a certain first-Baire-class function $f: C \to I^\omega$, i.e., $Y = \{(t, f(t)): t \in C\} \subset C \times I^\omega$.

Split the Cantor set $C$ into two disjoint Bernstein sets $C_1$ and $C_2$ (see [K, §40]), i.e., all compact subsets of $C_i$ are countable. Since $Y$ is strongly infinite-dimensional, either $Y_1 = \{(t, f(t)): t \in B_1\}$ or $Y_2 = \{(t, f(t)): t \in B_2\}$ is strongly infinite-dimensional. Suppose that $Y_2$ is strongly infinite-dimensional and let $X$ be the following subspace of $C \times C \times I^\omega$: $X = (B_1 \times T) \cup \{(t, t, f(t)): t \in B_2\} \subset C \times T$. Denote $X_1 = (B_1 \times T)$, $X_2 = \{(t, t, f(t)): t \in B_2\}$. The set $X_2$ is obviously homeomorphic to $Y_2$, hence it is strongly infinite dimensional. We will show that

\[(1) \quad X \text{ is a C-space.}\]

Suppose that $\mathcal{G}_1, \mathcal{G}_2, \ldots$ is a sequence of open covers of $X$. Fix an $i \geq 3$. Since the set $B_1 \times Z_{i-2}$ is 0-dimensional, there exists a family $\mathcal{W}_i$ of pairwise disjoint sets open in $C \times T$ such that $B_1 \times Z_{i-2} \subset \bigcup \mathcal{W}_i = W_i$ and for each $W \in \mathcal{W}_i$ the set $W \cap X$ is contained in some member of $\mathcal{G}_i$. Let $F = (B_1 \times T) \setminus \bigcup_{i=3}^{\infty} W_i$; we will show that $\dim F = 0$. Since $\bigcup_{i=3}^{\infty} W_i \supset B_1 \times Z$, $F$ is a closed subset of $C \times T$ contained in $B_1 \times Y$. Consider the projection $p_1: B_1 \times T \to B_1$. Since $T$ is compact, $p_1$, as well as $p = p_1/F: F \to B_1$, is a closed mapping. Moreover, for each $t \in B_1$ the set $p^{-1}(t) = (\{t\} \times T) \cap F$ is a compact set contained in $\{t\} \times Y$. It follows that $\dim p^{-1}(t) = 0$, because $Y$ is totally disconnected. Thus, by the Hure-
wicz Theorem (see [E, Theorem 1.12.4])
\[
\dim F \leq \sup \{ \dim p^{-1}(t) : t \in B_1 \} + \dim B_1 = 0.
\]

Now, let \( \mathcal{V}_2 \) be a family of pairwise disjoint open subsets of \( C \times T \) such that \( F \subset \bigcup_{\mathcal{V}_2} W_2 \) and for each \( W \in \mathcal{V}_2 \) there exists \( U \in \mathcal{G}_2 \) such that \( W \cap X \subset U \).

Put \( K = X \setminus \bigcup_{i=2}^{\infty} W_i \), since \( \bigcup_{i=2}^{\infty} W_i \supset B_1 \times T \), then \( K \subset X_2 \). We shall show that the set \( K \) is countable. Let \( p_2 : C \times T \to C \) be the projection. Since \( p_2 \) is a closed mapping, the set \( L = p_2((C \times T) \setminus \bigcup_{i=2}^{\infty} W_i) \) is closed in \( C \). Moreover, the set \( L \) is contained in \( B_2 \), hence it is countable. It follows that the set \( p_2(K) \subset L \) is countable, which implies that \( K \) is countable, because \( p_2|_{X_2} : X_2 \to B_2 \) is a one-to-one mapping.

Let \( \mathcal{U}_1 \) be a family of pairwise disjoint open subsets of \( X \) such that \( K \subset \bigcup_{\mathcal{U}_1} \) and each member of \( \mathcal{U}_1 \) is contained in some member of \( \mathcal{G}_1 \). Then the family \( \mathcal{U}_1, \mathcal{U}_2, \ldots \), where \( \mathcal{U}_i = \{ W \cap X : W \in \mathcal{U}_i \} \) for \( i \geq 2 \), satisfies conditions (i), (ii) and (iii).

Finally, let \( f : X \to C \) be the projection. Then \( f^{-1}(B_2) = X_2 \) and the Graph(\( f \mid X_2 \)) is homeomorphic to \( X_2 \) in a natural way, hence it is strongly infinite-dimensional. It follows by Lemma 2 that the product \( X \times B_2 \) is strongly infinite-dimensional. Since \( B_2 \) is a 0-dimensional metrizable separable space, it is homeomorphic to a subset \( B \) of the irrationals \( P \).

**Construction of Example 2.** In this example we use, among others, some ideas of [P2].

Let \( Z = \prod_{i=0}^{\infty} I_i \), where \( I_i = I \) for \( i = 0, 1, 2, \ldots \), be the Hilbert cube and the mappings \( p_0 : Z \to I_0 \) and \( p_n : \prod_{i=1}^{n} I_i \to \prod_{i=1}^{n} I_i \) be appropriate projections. Arrange all rational numbers in \( I_0 \) into a sequence \( q_1, q_2, \ldots \). Denote by \( T \) a compactum obtained from \( Z \) by attaching to each compactum \( \{ q_n \} \times \prod_{i=n+1}^{\infty} I_i \) the \( n \)-cube \( \prod_{i=n+1}^{\infty} I_i \), by the map \( p_n \), i.e., \( T \) is a quotient space defined by an upper semicontinous decomposition consisting of singletons \( \{ (t, x) \} \), where \( t \in P \) and \( x \in I_0 \), and the sets \( \{ q_n \} \times p_n^{-1}(y) \), where \( y \in \prod_{i=n+1}^{\infty} I_i \). Let \( \pi : Z \to T \) be the natural quotient mapping and \( X_1 = \pi(Q \times \prod_{i=1}^{\infty} I_i) \).

Using the Continuum Hypothesis, define a transfinite sequence \( G_1 \supset G_2 \supset \cdots \supset G_\omega \supset \cdots, \alpha < \omega_1 \), of \( G_\delta \)-subsets of \( T \) containing \( X_1 \) and such that for every \( G_\delta \)-set \( G \) in \( T \) containing \( X_1 \) there exists \( \alpha \) with \( G_\alpha \subset G \) (\( \omega_1 \) denotes the first uncountable ordinal). Again by CH, we can arrange all continua in \( Z \) intersecting \( \{ 0 \} \times \prod_{i=1}^{\infty} I_i \) and \( \{ 1 \} \times \prod_{i=1}^{\infty} I_i \) into a sequence \( K_1, K_2, \ldots, K_\alpha, \ldots, \alpha < \omega_1 \).

By transfinite induction we will choose, for each \( \alpha < \omega_1 \), a point \( y_\alpha \in K_\alpha \cap \pi^{-1}(G_\alpha) \cap p_0^{-1}(P) \) such that \( y_\alpha \notin \{ y_\beta : \beta < \alpha \} \). Suppose that \( \alpha = 0 \) or that we have already defined all points \( y_\beta \) for \( \beta < \alpha \). Since \( \pi^{-1}(G_\alpha) \) is a \( G_\delta \)-set in \( Z \) containing \( Q \times \prod_{i=1}^{\infty} I_i \), \( M = p_0(Z \setminus \pi^{-1}(G_\alpha)) \) is a \( F_\sigma \)-set in \( I_0 \), contained in \( P \) (notice that the projection \( p_0 \) is closed). Thus the set \( P \setminus M \) is of second category in \( I_0 \). Hence there exists \( p_a \in (P \setminus M) \setminus \{ p_0(y_\beta) : \beta < \alpha \} \). Since \( \{ p_a \} \times \prod_{i=1}^{\infty} I_i \subset \pi^{-1}(G_\alpha) \) and the continuum \( K_\alpha \) intersects \( \{ p_a \} \times \prod_{i=1}^{\infty} I_i \), there exists \( y_\alpha \in K_\alpha \cap \pi^{-1}(G_\alpha) \cap p_0^{-1}(p_a) \). If we put \( x_\alpha = \pi(y_\alpha) \) for every \( \alpha < \omega_1 \), then

\[
x_\alpha \in G_\alpha \cap \pi(K_\alpha) \setminus \{ X_1 \cup \{ x_\beta : \beta < \alpha \} \}.
\]
Claim. The subspace $X = X_1 \cup \{ x_\alpha : \alpha < \omega_1 \}$ of $T$ has the desired properties. We will show that

(2) $X$ is a C-space,

and

(3) the subspace $X_2 = \{ x_\alpha : \alpha < \omega_1 \}$ of $X$ is strongly infinite dimensional.

To show (2) suppose that $(x_1, x_2, \ldots)$ is a sequence of open covers of $X$. Decompose the set $X_1$ into a sequence $Z_1, Z_2, \ldots$ of 0-dimensional sets. For $i = 2, 3, \ldots$ choose a family $\mathcal{W}_i$ of disjoint open subsets of $T$ such that $Z_{i-1} \subset \mathcal{W}_i = \bigcup \mathcal{W}_i$ and for each $U \in \mathcal{W}_i$ there exists $G \in \mathcal{G}_i$ satisfying $U \cap X \subset G$ (see [E, Lemma 1.7.3]). Then $X_1 \subset \bigcup_{i=2}^{\infty} \mathcal{W}_i$ and there exists $\alpha$ such that $G_\alpha \subset \bigcup_{i=2}^{\infty} \mathcal{W}_i$. The set $X \setminus \bigcup_{i=2}^{\infty} \mathcal{W}_i \subset X \setminus G_\alpha$ is contained in $\{ x_\beta : \beta < \alpha \}$, hence it is countable. Thus there exists a family $\mathcal{N}_1$ of open subsets of $X$ satisfying conditions (i) and (ii) for $i = 1$ and such that $U_1 = \bigcup \mathcal{N}_1 \subset X \setminus \bigcup_{i=2}^{\infty} \mathcal{W}_i$. The families $\mathcal{W}_1, \mathcal{W}_2, \ldots$, where $\mathcal{W}_i = \{ W \cap X : W \in \mathcal{W}_i \}$ for $i \geq 2$, satisfy conditions (i), (ii), and (iii).

To prove (3) we will show that the subspace $\pi^{-1}(X_2)$ of the Hilbert cube, which is homeomorphic to $X_2$, is strongly infinite dimensional. For $n = 1, 2, \ldots$, let $A_n = \{ (x_i) \in \prod_{i=0}^{\infty} I_i : x_n = 0 \}$ and $B_n = \{ (x_i) \in \prod_{i=0}^{\infty} I_i : x_n = 1 \}$ be opposite faces of the Hilbert cube $Z$ and let $U_i$ and $V_i$ be open subsets of $Z$ such that $A_i \subset U_i$, $B_i \subset V_i$, and $U_i \cap V_i = \emptyset$. For $i = 1, 2, \ldots$, let $L_i$ be an arbitrary partition between $U_i \cap \pi^{-1}(X_2)$ and $V_i \cap \pi^{-1}(X_2)$ in $\pi^{-1}(X_2)$. The partition $L_i$ can be extended (see [E, Lemma 1.2.9]) to a partition $L'_i$ between $A_i$ and $B_i$ in $Z$ satisfying $L'_i \cap \pi^{-1}(X_2) \subset L_i$. Since $\cap_{i=1}^{\infty} L'_i$ contains a continuum connecting $\{0\} \times \prod_{i=0}^{\infty} I_i$ and $\{1\} \times \prod_{i=0}^{\infty} I_i$ (see [RSW, Lemma 5.2]), say $y_\alpha$, we have $y_\alpha \in \cap_{i=1}^{\infty} L'_i \cap \pi^{-1}(X_2)$. Therefore $\cap_{i=1}^{\infty} L_i \neq \emptyset$. This ends the proof of (3).

Finally, let $f : X \to I_0$ be the projection. Then $f^{-1}(P) = X_2$. Moreover, $\text{Graph}(f | X_2)$ is homeomorphic to $X_2$, hence it is strongly infinite-dimensional. Thus, by Lemma 2, the product $X \times P$ is strongly infinite-dimensional.

**References**


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