ON COMPARABILITY IN A TOPOS

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Dedicated to EJA

Abstract. While studying the category of "finite" sets associated to a nonstandard model of arithmetic [4], it became apparent that the law of trichotomy plays an important role in that context. The object of this note is to point out the strength of the obvious variants of trichotomy in a general topos. At it turns out even its mildest form (COMP) is quite restrictive, the usual variants (M-COMP, E-COMP) force the topos to be equivalent to a category of sets with AC.

Definition. A topos $E$ satisfies comparability (COMP) if for any two objects $A$, $B$ there is a morphism $A \to B$ or $B \to A$.

Observations. a. If $E$ satisfies COMP, so does any full subcategory of $E$, in particular $SH_j(E)$ for some topology $j$, and $E_G$, the topos of coalgebras for an idempotent left exact cotriple $G$.

b. Any filtered colimit of toposes with COMP has COMP.

c. In a topos with COMP the lattice of subobjects of 1 is totally ordered.

d. COMP is not stable ($S$, the category of sets with AC, has COMP, but $S/A$ doesn’t).

Proposition 1. If supports split in $E$ and the subobjects of 1 are totally ordered, then $E$ has COMP.

Proof. Given $A$, $B$ we have epimorphisms to their supports $A \to \sigma(A)$, $B \to \sigma(B)$, also their splittings going in the opposite direction, as well as, say $\sigma(A) \to \sigma(B)$, thus by composition, a morphism from $A$ to $B$.

Examples. It is well known that if $S$ satisfies AC then COMP is true in $S$. For any well order $T$, $S^T$ has COMP. Let $M$ be the monoid consisting of $e$ and a single idempotent element $a$, then $S^M^\text{op}$ has COMP. In both cases the conditions of Proposition 1 are fulfilled. Let $G$ be a group whose lattice of subgroups divided by the conjugation relation is a total order (e.g. $Z/(p^n)$); then $S^G$ satisfies COMP.

Proposition 2. If $E$ is a Grothendieck topos and has COMP, then $E$ is connected and locally connected (molecular).

Proof. Indeed by c, there are no nontrivial complemented subobjects of 1, i.e., $E$ is connected. Let $C$ be a generating full subcategory of $E$; then by Giraud's theorem...
we have $E = Sh_{can}(C)$. The canonical sieves are nonempty and COMP insures that they are connected. Hence $(C, \text{can})$ is a molecular site (see [1]). □

**Definition.** A topos $E$ satisfies **mono-comparability** (M-COMP) if for any two objects $A, B$ there is a monomorphism $A \to B$ or $B \to A$.

**Proposition 3.** If a topos satisfies M-COMP, then supports split.

**Proof.** Let $p: A \to \sigma(A)$ be the support morphism. If there is a monomorphism $m: A \to \sigma(A)$, then the unicity of morphisms to $\sigma(A)$ implies that $p = m$. Hence it has an inverse. On the other hand, if there is a monomorphism $m: \sigma(A) \to A$, then, again by the same unicity, we must have $pm = \text{id}_{\sigma(A)}$. □

**Proposition 4.** In an M-COMP topos the terminal object 1 has no proper subobjects.

**Proof.** Let $U \to 1$ be a subobject; then if the unique morphism $!: U + U \to 1$ is a monomorphism, then from $i_1 = i_2$ it follows that $i_1 = i_2$ hence $U = \text{eq}(i_1, i_2) \approx 0$. Otherwise a monomorphism $1 \to U + U$ implies $1 \approx \sigma(U + U) \approx \sigma(U) \approx U$. □

**Proposition 5.** An M-COMP topos is Boolean.

**Proof.** By Proposition 4 the only morphisms $1 \to \Omega$ are true and false. This implies that there is no monomorphism from $1 + 1 + 1$ to $\Omega$. So there must be a monomorphism $\Omega \to 1 + 1 + 1$ which splits $\Omega$ into three summands $\Omega \approx T + F + X$. $T$ and $F$ are respectively true and false because we have in any topos the canonical monomorphism $1 + 1 \to \Omega$. Now, if $X$ is not isomorphic to 0, then the morphism $X \to 1$ must be an epimorphism, and by Proposition 3 it has a section $1 \to X$ which provides a third morphism from 1 to $\Omega$, contradicting the first statement of this proof. □

**Proposition 6.** In a topos with M-COMP 1 is a generator.

**Proof.** Let $u, v: x \Rightarrow y$ be any two morphisms and let $E$ be their equalizer, which by Proposition 5 admits a complement $E'$. If $E'$ is not 0, then, as above, there is a morphism $x: 1 \to E'$, whence $ux \neq vx$. □

**Definition.** A topos $E$ satisfies **epi-comparability** (E-COMP) if for any objects $A, B \neq 0$ there exists an epimorphism $A \to B$ or $B \to A$.

**Proposition 7.** In an E-COMP topos the terminal object 1 has no proper subobjects.

**Proof.** Let $U \to 1$ be a subobject and, again, one looks at $U + U$. If the unique morphism $U + U \to 1$ is an epimorphism, then so must be $U \to 1$ (since $\sigma(U + U) = \sigma(U)$), hence it is an isomorphism. If there is an epimorphism $1 \to U + U$, then passing to supports yields $1 = \sigma(1) \to \sigma(U + U) \approx \sigma(U) \approx (U)$. □

The following is a topos version of a celebrated result due to G. Cantor:

**Lemma 8.** Let $(\Delta, \Gamma): E \to S$ be any topos defined over sets and $E \in E$. Then there are no epimorphisms $E \to \Delta \Gamma(2^E)$. 

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Proof. Assume that there exists an epimorphism \( p: E \to \Delta \Gamma(2^E) \) and use it to form the morphism:

\[
E \to \Delta \Gamma(2^E) \times E \to 2^E \times E \to 2
\]

To it corresponds a complemented subobject \( m: E' \to E \) with characteristic morphism \( \chi \); let \( \varphi = -\chi \), with associated subobject \( C_E E' \), and \( \Gamma(\varphi): 1 \to 2^E \) its exponential transpose. Applying \( \Delta \Gamma \), one obtains the morphism \( \Delta \Gamma(\varphi): 1 \to \Delta \Gamma(2^E) \) satisfying the usual \( \epsilon \Delta \Gamma(\varphi) = \Gamma(\varphi) \). Pulling back along \( pm \) one obtains:

\[
\begin{align*}
F' & \to F & \to 1 \\
\downarrow n' & \downarrow n & \downarrow \Delta \Gamma(\varphi) \\
E' & \to E & \to \Delta \Gamma(2^E)
\end{align*}
\]

Note that \( F \neq 0 \). From the commutativity of the rightmost square one obtains, after composing with \( ev(e \times E) \), two equal morphisms. On one hand, by unraveling the definition of \( \Delta \Gamma(\varphi) \), one has a morphism \( F \times E \to E \to 2 \), whose associated subobject is: \( F \times C_E E' \to F \times E \). The pull-back of it along \( (F, n) \) is \( C_E F' \). On the other hand, one has a morphism which composed with \( (F, n) \) yields the commutative diagram:

\[
\begin{align*}
F' & \to F & \to 1 \\
\downarrow n' & \downarrow n & \downarrow \Delta \Gamma(\varphi) \\
E' & \to E & \to \Delta \Gamma(2^E)
\end{align*}
\]

The associated subobject of \( F \) is none other than \( F' \). Therefore \( C_E F' = F' \), a contradiction since \( F \neq 0 \). □

Corollary 9. For any topos \( E \) and for any \( E \in E \) there exists \( S \in S \) such that there are no epimorphisms from \( E \) to \( \Delta(S) \). □

Proposition 10. An \( E \)-COMP topos is Boolean.

Proof. Consider \( \Omega \) and \( \Delta \Omega \), for arbitrarily large \( S \). The preceding corollary implies the existence of an epimorphism \( p: \Delta \Omega \to \Omega \). As there are only two morphisms from \( 1 = \Delta 1 \) to \( \Omega \), \( p \) must factor through the canonical monomorphism \( \Delta 2 = 1 + 1 \to \Omega \), which forces it to be an epimorphism as well, hence an isomorphism. □

Proposition 11. In a topos with \( E \)-COMP \( 1 \) is a generator.

Proof. First, \( E \) is molecular, so let \( G \) be a generator which is a molecule as well. As before, there is \( S \in S \) and an epimorphism \( p: \Delta S \to G \). Any element \( s: 1 \to S \) induces a monomorphism \( p\Delta(s): 1 \to G \) which has a complement \( G' \), \( G \) being a molecule forces \( G' = 0 \). Thus \( p\Delta(s) \) is an isomorphism. □

In view of their similar consequences, one suspects that \( M \)-COMP and \( E \)-COMP are equivalent for any topos (they are in set theory [6]). Indeed, this is the case, but
these consequences are used in the present proof. First, let us take a diversion through well-ordered objects in a Boolean topos in which $1$ is a generator. In such a topos $\Omega = 2$ and let $P(A) = 2^A$ and $P_*(A) = P(A) - \{\text{false}\}$, i.e., the object of nonempty subobjects of $A$.

Given a partial order $\leq \Rightarrow A \times A$ one derives a morphism $[\uparrow]: A \rightarrow P(A)$, $[a \uparrow] = \{x: a \leq x\}$, which factors through $P_*(A)$.

**Definition.** $(A, \leq)$ is well ordered if there exists a morphism $\min: P_*(A) \rightarrow A$ such that (i) $\min[a \uparrow] = a$, (ii) $X \subset [\min(X) \uparrow]$, and (iii) $\min(X) \in X$.

**Lemma 12.** Let $A$ be a well-ordered object and $B \subset A$ an initial segment. Then there is $b \in A$ such that $B = (b \downarrow) = \{x: x < b\}$.

**Proof.** Set $b = \min(C Abed B)$. ■

**Lemma 13.** Given two well-ordered objects $(A, \leq), (B, \leq)$, there is an order-preserving monomorphism $m: A \rightarrow B$, or $n: B \rightarrow A$, which is initial.

**Proof (adapted from Zermelo via [2]).** Let $\Phi$ be the object of elements $\varphi \subset P(A \times B)$ that satisfy the conditions: (i) $\emptyset \in \varphi$, (ii) $\varphi' \subset \varphi$ implies $\bigcup \varphi' \in \varphi$, (iii) $X \in \varphi$ and $p_1X \neq A$ and $p_2X \neq B$ implies $X^* \in \varphi$, where $X^* = X \cup \{\min(A - p_1X)$, $\min(B - p_2X)\}$. Note that $X \subset X^*$. Let $\varphi_0 = \cap(\varphi: \varphi \in \Phi)$ and $C = \bigcup(\{X: X \in \varphi_0\})$. Thus $C \subset A \times B$ and $(a', b), (a'', b') \in C$ implies $a' = a''$, as well as $(a', b'), (a, b'') \in C$ implies $b' = b''$. Indeed, say $a' < a''$ and let $\varphi' = \{X \in \varphi_0: (a'', b') \notin X\}$, $\varphi'$ satisfies (i) and (ii) trivially and $\varphi' \subset \varphi_0$. If $(a', b) \in X$, then $\min(B - p_2X) > b$, thus $(a'', b) \notin X^*$; if $(a', b) \notin X$ then $\min(A - p_1X) < a'$ so that $(a'', b) \notin X^*$, i.e., in both cases $X^* \notin \varphi'$.

Hartogs' operator $(\_)^+$ can be defined as well. Given an object $X$ let $X^- = \{(A, \leq): A \subset X, \leq \text{ is a w.o. on } A\}$ and define $(A', \leq ') \leq_1 (A'', \leq'')$ to mean the existence of an order-preserving monomorphism $m: A' \rightarrow A''$ such that the $\text{Im}(m)$ is an initial segment of $A''$. $(A', \leq ') \equiv (A'', \leq'')$ means that $m$ is an isomorphism. Let $X^+ = X^- / \equiv$, then $\leq_1$ induces a well order on $X^+$. For any nonempty family $\{B_j\}_{j \in J}$ of elements of $X^+$, choose any $k \in J$ and a representative $A_k$ for $B_k$. Lemma 13 says that the family $\{B_j\}_{j \in J}$ is totally ordered so one can ignore those $B_j$ with $B_j > B_k$. For each $j$ with $B_j \leq B_k$ let $A_j$ be the initial segment of $A_k$ provided by Lemma 13. In turn, for each $A_j$ one has elements $a_j$ given by Lemma 12, forming a subobject of $A_k$ whose min is $a_0$. Then the corresponding $A_0$ and $B_0$ are minimal as well; therefore $X^+$ is well ordered.
Theorem 14. The following are equivalent:
(a) \( E \) satisfies M-COMP.
(b) \( E \) satisfies E-COMP.
(c) \( E \) satisfies AC and 1 is a generator.

Proof. (a) \( \Rightarrow \) (b) Let \( A, B \neq 0 \); by hypothesis there is a monomorphism, say \( m: A \to B \); \( A \neq 0 \) implies that there is a morphism \( a: 1 \to A \), so one can define an epimorphism \( p: B \to A \) by cases \( p(b) = m^{-1}(b) \) for \( b \in \text{Im}(m) \) and \( p(b) = a \) for \( b \notin \text{Im}(m) \).

(b) \( \Rightarrow \) (c) Once the Hartogs' operator is definable, the proof in [6, p. 10] is valid in a Boolean topos in which 1 generates.

(c) \( \Rightarrow \) (a) It is known that AC forces the topos \( E \) to be Boolean [3], so that the proof outlined in [2, ex. 4, p. 201] is completely in \( E \). ■

References

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