A PROPERTY OF IDEALS IN POLYNOMIAL RINGS

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Abstract. Every ideal in the polynomial ring in \( n \) variables over an infinite field has a reduction generated by \( n \) elements.

Eisenbud and Evans [2] proved that every ideal in \( k[X_1, \ldots, X_n] \) can be generated up to radical by \( n \) elements (where \( k \) is a field). Avinash Sathaye [7] and Mohan Kumar [5] proved a locally complete intersection in \( k[X_1, \ldots, X_n] \) can be generated by \( n \) elements.

In this short note we show that every ideal in \( k[X_1, \ldots, X_n] \) has a nice approximation generated by \( n \) elements. More precisely, we prove the following.

Theorem. Let \( k \) be an infinite field. Then every ideal \( I \) in \( k[X_1, \ldots, X_n] \) has a reduction \( J \) generated by \( n \) elements.

By [6], \( J \) is a reduction of \( I \) if there exists an integer \( r \) such that \( JJ^r = I^{r+1} \). Northcott and Rees [6] point out that \( J \) can be regarded as a simplified version of \( I \) preserving many properties of \( I \), in particular the multiplicities at minimal prime over-ideals. Moreover, \( J \) has the same radical as \( I \) and if \( I \) is locally a complete intersection, then it is the only reduction of itself, hence a connection between our result and those of Eisenbud and Evans and Sathaye and Kumar.

Proof of the theorem. Since \( k[X_1, \ldots, X_n] \) is a UFD, we can assume that \( n \geq \text{dim}(A/I) + 2 \). Let \( g_1, \ldots, g_r \) be a system of generators of \( I \). Set \( A = k[X_1, \ldots, X_n] \) and \( B = k[g_1, \ldots, g_r] \subset A \). The dimension of \( B \) is at most \( n \), since its quotient field is a subfield of \( k(X_1, \ldots, X_n) \) and therefore has transcendence degree \( \leq n \). Denote by \( P \) the ideal of \( B \) generated by \( g_1, \ldots, g_r \). Let \( \overline{P} \) be the completion of \( P \) in \( B_P \), the localization of \( B \) at \( P \) (i.e. \( \overline{P} \) is the maximal ideal of \( B_P \)). Since \( B_P \) is local of dimension \( \leq n \), by Burch [1] \( \overline{P} \) has a reduction \( \overline{Q} \) generated by \( n \) elements \( h_1, \ldots, h_n \). Let \( r \) be an integer such that \( \overline{Q} \overline{P}^r = \overline{P}^{r+1} \). Since \( \overline{Q} \) is \( \overline{P} \)-primary, there exists a unique \( P \)-primary ideal \( Q \subset B \), the image of which in \( B_P \) coincides with \( \overline{Q} \). Since \( P \) is a maximal ideal of \( B \), we see that the ideals \( Q \overline{P}^r \) and \( P^{r+1} \) are \( P \)-primary and their images in \( B_P \) coincide. Therefore, \( Q \overline{P}^r = P^{r+1} \) in \( B \), since there is a one-to-one correspondence between the \( \overline{P} \)-primary ideals of \( B_P \) and the \( P \)-primary ideals of \( B \).

Let \( J \) be the extension of \( Q \) in \( A \). Since the extension of \( P \) is \( I \) and \( Q \overline{P}^r = P^{r+1} \), we see that \( JJ^r = I^{r+1} \), i.e. \( J \) is a reduction of \( I \).
Let \( s \in B \setminus P \) be an element such that \( Q_s = (h_1, \ldots, h_n) \). Thus \( J_s = (h_1, \ldots, h_n) \).

Since \( Q \) is \( P \)-primary and \( P \) is a maximal ideal of \( B \), we see that \( (s) + Q = (1) \), i.e., regarding \( s \) as an element of \( A \), that \( (s) + J = (1) \). The conditions \( (s) + J = (1) \) and \( J_s = (h_1, \ldots, h_n) \) imply that \( J/J^2 \) is generated by \( n \) elements \( h_1, \ldots, h_n \). Now Theorem 5 of Mohan Kumar [4] tells us that \( J \) is generated by \( n \) elements, since \( n \geq \dim(A/J) + 2 \). Q.E.D.

Remarks. 1. In a similar way one can prove that every ideal in a finitely generated \( n \)-dimensional algebra over an infinite field has a reduction generated by \( n + 1 \) elements.

2. For every prime ideal \( P \subset B \) containing \( I \) we have the dimension of

\[
\frac{(J + P \cdot I)}{P \cdot I}
\]

as a vector space over the quotient field of \( A/P \) is \( \geq \) height \( I \). This improves Eisenbud and Evans [2, 3] who proved only that \( J \subset P \cdot I \).

3. Some generalization of our theorem is possible to the case \( I \subset A[X] \), where \( A \) is an \((n - 1)\)-dimensional finitely generated algebra over an infinite field. For this one has to use S. Mandall's extension of Mohan Kumar's theorem to ideals in \( A[X] \) (cf. [5]).

Conjecture. Let \( A \) be a commutative Noetherian ring of dimension \( n - 1 \) such that the residue field of every maximal ideal of \( A \) is infinite. Let \( I \) be an ideal of \( A \) or \( A[X] \). Then \( I \) has a reduction generated by \( n \) elements.

References


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