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CONVERGENCE OF CARDINAL SERIES

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ABSTRACT. The result of this paper is a generalization of our characterization of the limits of multivariate cardinal splines. Let M_n denote the n-fold convolution of a compactly supported function $M \in L_2(\mathbf{R}^d)$ and denote by

$$S_n := \left\{ \sum_{j \in \mathbf{Z}^d} c(j) M_n(\cdot - j) \colon c \in l_2(\mathbf{Z}^d) \right\}$$

the span of the translates of M_n. We prove that there exists a set Ω with $\operatorname{vol}_d(\Omega) = (2\pi)^d$ such that for any $f \in L_2(\mathbf{R}^d)$,

$$\operatorname{dist}(f, S_n) \to 0 \quad \text{as } n \to \infty,$$

if and only if the support of the Fourier transform of f is contained in $\overline{\Omega}$.

We extract the essential features of our earlier arguments [**1–4**] concerning the limits of box-splines as their degree tends to infinity. Somewhat surprisingly, the resulting discussion, although covering a more general situation, is very much shorter.

We start with a compactly supported (nonzero) L_2-function M on $\mathbf{R}^d$ for which the Fourier transform

$$\hat{M}(\xi) := \int M(x) \exp(-ix\xi)\, dx$$

satisfies

$$|\hat{M}(\xi)| = O(|\xi|^{-1}), \qquad |\xi| \to \infty. \tag{1}$$

With $M_n := M * \cdots * M$ denoting the n-fold convolution of M, we consider approximation in L_2 from the span

$$S_n := \left\{ \sum_{j \in \mathbf{Z}^d} c(j) M_n(\cdot - j) \colon c \in l_2(\mathbf{Z}^d) \right\}$$

of the integer translates of M_n. We wish to characterize the class

$$S_\infty := \left\{ f \in L_2(\mathbf{R}^d) \colon \lim_{n \to \infty} \operatorname{dist}(f, S_n) = 0 \right\}.$$

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For this we introduce the set

$$\Omega := \{\xi \in \mathbf{R}^d : |\hat{M}(\xi + 2\pi j)| < |\hat{M}(\xi)|,\ j \in \mathbf{Z}^d \setminus 0\}$$

and establish the following

PROPOSITION. *Ω is a fundamental domain, i.e.*

$$\overline{\Omega} \cap (\Omega + 2\pi j) = \emptyset, \qquad j \neq 0;$$

$$\operatorname{meas}\left(\mathbf{R}^d \setminus \bigcup_j (\Omega + 2\pi j)\right) = 0.$$

PROOF. To prove the first assertion, let $\xi = \lim_{\nu\to\infty} \xi_\nu$ with $\xi_\nu \in \Omega$. Then the assumption $\xi - 2\pi j \in \Omega$ with $j \neq 0$ leads to the contradiction

$$\begin{aligned} 1 &> |\hat{M}((\xi - 2\pi j) + 2\pi j)/\hat{M}(\xi - 2\pi j)| \\ &= \lim_{\nu\to\infty} |\hat{M}(\xi_\nu)/\hat{M}(\xi_\nu - 2\pi j)| \geq 1. \end{aligned}$$

For the second assertion, consider the function

$$j \mapsto |\hat{M}(\xi + 2\pi j)|^2.$$

If this function has a unique maximum, say at $j = j_*$, then $\xi \in \Omega + 2\pi j_*$. Thus the complement of $\bigcup_j (\Omega + 2\pi j)$ is contained in $\{\xi : f_j(\xi) = f_k(\xi), \text{ some } j \neq k\}$, where $f_j := |\hat{M}(\cdot + 2\pi j)|^2$. Since the zero set of a nontrivial real analytic function is of measure zero, it remains to show that $f_j - f_k$ cannot vanish identically. But this follows since (1) cannot hold for a function which is periodic in the direction $j - k \neq 0$.

THEOREM. *$f \in S_\infty$ if and only if the support of $\hat{f}$ is contained in $\overline{\Omega}$.*

PROOF. For

$$\xi \in D := \{\xi \in \mathbf{R}^d : \hat{M}(\xi) \neq 0\} \supset \Omega,$$

we define

$$a_j(\xi) := \hat{M}(\xi + 2\pi j)/\hat{M}(\xi)$$

and introduce the trigonometric polynomial

$$\begin{aligned} P_n(\xi) &:= \sum_j M_n(j)\exp(-ij\xi) = \sum_j \hat{M}_n(\xi + 2\pi j) \\ &= \hat{M}_n(\xi)\sum_j (a_j(\xi))^n \end{aligned}$$

with the last equality holding, at least, on D. Let $\xi \in \Omega$. For any $j \neq 0$ and $\xi \in \Omega$,

$$|a_j(\xi)| \leq 1 - \varepsilon(j, \xi) \tag{2.1}$$

for some positive $\varepsilon(j, \xi)$, while, by (1), there exists a positive constant C such that for all but finitely many j,

$$|a_j(\xi)| \leq 1/(1 + C|j|). \tag{2.2}$$

Consequently, for $\xi \in \Omega$,

$$P_n(\xi)/\hat{M}_n(\xi) = \sum_j (a_j(\xi))^n \to 1, \qquad n \to \infty, \tag{3}$$

and the convergence is uniform on compact subsets Ω_1 of Ω. This shows, in particular, that, for large enough n, P_n does not vanish on such Ω_1.

(i) Assume that $f \in L_2$ and $\hat{f}$ vanishes a.e. outside $\overline{\Omega}$. Denote by χ the characteristic function of a compact subset Ω_1 of Ω. Since Ω is a fundamental domain, we can expand $\hat{f}\chi/P_n$ in a Fourier series

$$(\hat{f}\chi/P_n)(\xi) =: \sum_j c_n(j)\exp(ij\xi), \qquad \xi \in \Omega,$$

with coefficients $c_n \in l_2$. This implies that

$$s_n := \sum_j c_n(j) M_n(\cdot - j) \in L_2.$$

Since $\hat{f}$ vanishes a.e. outside $\overline{\Omega}$ and, by the Proposition, $\overline{\Omega} \setminus \Omega$ has measure zero,

$$|\hat{f} - \hat{s}_n|^2_{L_2(\mathbf{R}^d)} = |\hat{f} - \hat{s}_n|^2_{L_2(\Omega)} + \sum_{j \neq 0} |\hat{s}_n(\cdot + 2\pi j)|^2_{L_2(\Omega)}.$$

The first term is estimated by

$$|\hat{f} - \hat{s}_n|_{L_2(\Omega)} \leq |\hat{f} - \chi\hat{f}|_{L_2(\Omega)} + |\chi\hat{f} - \chi\hat{f}\hat{M}_n/P_n|_{L_2(\Omega)},$$

where the first norm on the right-hand side is small if Ω_1 is chosen close to Ω, while, for fixed Ω_1, the second norm is small by (3) if n is sufficiently large. The jth term in the sum is the square of

$$\begin{aligned} |\hat{M}_n(\cdot + 2\pi j)(\hat{f}\chi/P_n)|_{L_2(\Omega)} &= |(a_j)^n \hat{M}_n(\hat{f}\chi/P_n)|_{L_2(\Omega)} \\ &\leq (|a_j|_{L_\infty(\Omega_1)})^n |\hat{M}_n/P_n|_{L_\infty(\Omega)} |\hat{f}|_{L_2(\Omega_1)}. \end{aligned}$$

By (2.$*$) this implies that the sum is small for large n.

(ii) Assume that $s_n = \sum_j c_n(j) M_n(\cdot - j)$ converges to f in L_2. Since

$$\hat{s}_n(\xi + 2\pi j) = (a_j(\xi))^n \hat{s}_n(\xi), \qquad \xi \in D,$$

we see from (2.$*$) that for $j \neq 0$

$$|\hat{s}_n|_{L_2(\Omega_1 + 2\pi j)} \leq (|a_j|_{L_\infty(\Omega_1)})^n |\hat{s}_n|_{L_2} \to 0$$

for any compact subset Ω_1 of Ω. Since $\mathbf{R}^d \setminus \bigcup_j (\Omega + 2\pi j)$ has measure zero, it follows that, as an element of L_2, $\hat{f}$ vanishes outside $\overline{\Omega}$.

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