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POINTWISE ESTIMATES FOR CONVEX POLYNOMIAL APPROXIMATION

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ABSTRACT. For a convex function $f \in C[-1,1]$ we construct a sequence of convex polynomials p_n of degree not exceeding n such that $|f(x) - p_n(x)| \leq C\omega_2(f, \sqrt{1-x^2}/n)$, $-1 \leq x \leq 1$. If in addition f is monotone it follows that the polynomials are also monotone thus providing simultaneous monotone and convex approximation.

1. Introduction and main results. In recent years there has been extensive research on monotone polynomial approximation. Lorentz and Zeller [**8**], Lorentz [**7**] and finally DeVore [**1**] have obtained Jackson type estimates on the rate of uniform approximation of monotone functions by monotone polynomials. Their results can be summarized as follows. For a monotone $f \in C^k[-1,1]$ there are monotone polynomials p_n of degree not exceeding n such that

$$\|f - p_n\| \leq Cn^{-k}\omega(f^{(k)}, n^{-1}), \tag{1}$$

where here and in the sequel C is an absolute constant independent of f and n; and $\omega(g,\cdot)$ is the usual modulus of continuity of g.

Recently Shvedov [**10**] has extended these results by showing that for a monotone $f \in C[-1,1]$ there are monotone polynomials p_n of degree $\leq n$ such that

$$\|f - p_n\| \leq C\omega_2(f, 1/n) \tag{2}$$

where $\omega_2(f,\cdot)$ is the second modulus of smoothness of f. Moreover, he has proved that one cannot expect (2) to hold with ω_3 replacing ω_2.

Shvedov [**9, 10**] also discussed the question of convex polynomial approximation to convex functions $f \in C[-1,1]$ showing that there exist convex polynomials p_n satisfying (2).

Recently DeVore and Yu [**2**] have constructed a sequence of monotone polynomials p_n associated with a monotone function $f \in C[-1,1]$ and yielding the following pointwise approximation rate:

$$|f(x) - p_n(x)| \leq C\omega_2(f, \sqrt{1-x^2}/n), \qquad -1 \leq x \leq 1. \tag{3}$$

Denote $\varphi(x) = \sqrt{1-x^2}$ and define

$$\omega_2^\varphi(f,\delta) = \sup_{\substack{0\leq h\leq\delta \\ -1\leq x\leq 1}} |\Delta^2_{h\varphi(x)} f(x)|$$

where

$$\Delta^2_{h\varphi(x)} f(x) = f(x - h\varphi(x)) - 2f(x) + f(x + h\varphi(x))$$

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if $x \pm h\varphi(x) \in [-1,1]$ and $= 0$ elsewhere. Then the author [**6**] has shown that the DeVore and Yu polynomials [**2**] also yield uniform monotone approximation at the rate

$$\|f - p_n\| \le C\omega_2^{\varphi}(f, 1/n) \tag{4}$$

which is an improvement of (1), (2) and even (3) in certain cases and which, although being a uniform estimate, takes into account the behavior of f near the endpoints of the interval allowing rougher behavior of f near the endpoints than in the middle of the interval.

The ω_2^{φ} modulus of smoothness has recently been used extensively by Ditzian [**3**, **4**] and Totik [**11**] and in a yet unpublished paper by Ditzian and Totik [**5**]. It is equivalent to the Peetre functional

$$K_2(f,t) = \inf\{\|f - g\|_\infty + t^2\|(1-x^2)g''(x)\|_\infty\}$$

where the infimum is taken over all $g \in C^1[-1,1]$ such that g' is locally absolutely continuous in $[-1,1]$ and $(1-x^2)g''(x) \in L_\infty[-1,1]$.

In this note we will modify the DeVore-Yu polynomials and for a convex f, obtain polynomials which are convex on $[-1,1]$ and which satisfy (3) and (4), thus improving Shvedov's estimates (2). Also, if in addition, f is monotone, then the polynomials are monotone. Our result is therefore

THEOREM 1. *There exists an absolute constant C such that for any convex function $f \in C[-1,1]$ and every $n \ge 1$, there is a convex polynomial p_n of degree not exceeding n satisfying* (3) *and* (4). *If in addition f is monotone, then so is p_n.*

The proof is quite constructive and is the contents of §2.

Classical converse theorems for algebraic polynomials and Theorem 1 allow us to characterize the convex functions in $\mathrm{Lip}^*\,\alpha$, $0 < \alpha < 2$, by means of their rate of approximation by algebraic polynomials. Since $\mathrm{Lip}^*\,\alpha$, $0 < \alpha < 2$, is defined as the space of all functions f such that $\omega_2(f,t) = O(t^\alpha)$, we have

THEOREM 2. *For $0 < \alpha < 2$, a function f is convex and in $\mathrm{Lip}^*\,\alpha$ if and only if for each $n \ge 1$ there is a convex polynomial p_n such that*

$$|f(x) - p_n(x)| \le C(\sqrt{1-x^2}/n)^\alpha, \qquad -1 \le x \le 1.$$

2. Proofs. Following [**2**] we approximate f by a piecewise linear function S_n which interpolates f at certain points $-1 = \xi_0 < \xi_1 < \cdots < \xi_n = 1$. However, these are fewer points than in [**2**], in fact, a subset of the points there. The choice of the ξ_j's is made in the following way. Let $J_n(t)$ denote the Jackson kernel

$$J_n(t) = \lambda_n \left(\frac{\sin nt/2}{\sin t/2}\right)^8, \qquad \int_{-\pi}^{\pi} J_n(t)\,dt = 1,$$

and define

$$T_j(t) = \int_{t-t_j}^{t+t_j} J_n(u)\,du, \qquad j = 0, \ldots, n,$$

where $t_j = j\pi/n$, $j = 0, \ldots, n$. Now for $x = \cos t$ let $r_j(x) = T_{n-j}(t)$ and define

$$R_j(x) = \int_{-1}^{x} r_j(u)\,du, \qquad j = 0, \ldots, n.$$

Note that since $T_0 \equiv 0$ and $T_n \equiv 1$ we have $R_0(x) = 1 + x$ and $R_n(x) \equiv 0$. The points ξ_j are defined by the equations $1 - \xi_j = R_j(1)$.

Like in [**2**] it follows by the definition of T_n that $T_{n-j} - T_{n-(j+1)} \geq 0$. Hence $r_j - r_{j+1} \geq 0$ and so $R_j - R_{j+1}$ is increasing for $j = 0, 1, \ldots, n-1$. Therefore $-1 = \xi_0 < \cdots < \xi_n = 1$. Although our ξ_j's are only a subset of their counterparts in [**2**] (our ξ_j corresponds to ξ_{2j-n}, $j = 0, \ldots, n$ in [**2**]) they nevertheless have similar distribution in $[-1, 1]$. This can be summarized in the following lemma (see [**2** or **6**]).

LEMMA A. *Let* $\delta_j = (\sin t_{n-j})/n + 1/n^2$, $j = 0, \ldots, n$. *Then*
(i) $C_0\delta_j \leq \xi_{j+1} - \xi_j \leq C_1\delta_j$, $j = 0, \ldots, n-1$,
(ii) $C_0\delta_j \leq \delta_{j+1} \leq C_1\delta_j$, $j = 0, \ldots, n-1$,
(iii) *for any* $\xi_j \leq u \leq \xi_{j+1}$, $1 \leq j \leq n-2$,

$$\xi_{j+1} - \xi_j \leq C\sqrt{1-u^2}/n.$$

Now the piecewise linear interpolant S_n in $[\xi_j, \xi_{j+1}]$ has the slope

$$s_j = \frac{f(\xi_{j+1}) - f(\xi_j)}{\xi_{j+1} - \xi_j}, \qquad j = 0, \ldots, n-1,$$

and if $\varphi_j = (x - \xi_j)_+$ we can write

$$S_n(x) = f(-1) + s_0(1+x) + \sum_{j=1}^{n-1}(s_j - s_{j-1})\varphi_j(x).$$

Replacing $\varphi_j(x)$ by a sufficiently good approximation to it, namely, $R_j(x)$, brings us to the polynomials

$$\begin{aligned} L_n(f) &= f(-1) + s_0R_0 + \sum_{j=1}^{n-1}(s_j - s_{j-1})R_j \\ &= f(-1) + \sum_{j=0}^{n-1} s_j(R_j - R_{j+1}). \end{aligned} \tag{5}$$

It has already been mentioned that $R_j - R_{j+1}$ is increasing so that $L_n(f)$ is monotone if all s_j have the same sign which is the case when f is monotone. Also it follows by Lemma A and the proof in [**2**] that

$$|f(x) - L_n(f)(x)| \leq C\omega_2(f, \sqrt{1-x^2}/n) \tag{6}$$

and by the proof in [**6**] we have

$$\|f - L_n(f)\| \leq C\omega_2^{\varphi}(f, 1/n). \tag{7}$$

Thus our proof is complete if we show that $L_n(f)$ is convex when f is. Now when f is convex then $s_j - s_{j-1} \geq 0$ for $j = 1, \ldots, n-1$; also $R_0(x) = 1 + x$ is linear. Thus it follows by the first equation in (5) that $L_n(f)$ is convex if $R_j'' \geq 0$ for

$j = 1, \ldots, n-1$. To prove this we see that for $x = \cos t$, $0 < t < \pi$,

$$\begin{aligned} R_j''(x) = r_j'(x) &= \frac{dt}{dx}\frac{d}{dt}\int_{t-t_{n-j}}^{t+t_{n-j}} J_n(u)\,du \\ &= \frac{-1}{\sin t}\left[J_n\left(t + \frac{(n-j)\pi}{n}\right) - J_n\left(t - \frac{(n-j)\pi}{n}\right)\right] \\ &= \frac{-\lambda_n}{\sin t}\sin^8 \frac{nt + (n-j)\pi}{2} \\ &\quad \times \left[\frac{1}{\sin^8 \frac{1}{2}(t + (n-j)\pi/n)} - \frac{1}{\sin^8 \frac{1}{2}(t - (n-j)\pi/n)}\right] \\ &\geq 0 \end{aligned}$$

due to the inequality

$$\sin(\alpha + \beta) \geq |\sin(\alpha - \beta)|, \quad \text{if } 0 \leq \alpha, \beta \leq \pi/2.$$

This was observed by Lorentz and Zeller [8] who used it to obtain estimates on monotone approximation. Surprisingly we apply it to ascertain that the polynomials we have are convex.

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