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INDICES AND NULLITIES OF YANG-MILLS FIELDS

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ABSTRACT. In this note, we give an upper estimate of index and nullity of every Yang-Mills field R on every principal bundle P with structure group G over a Riemannian manifold (M, g) with Ricci tensor $\mathrm{Ric} \geq kg$, $k > 0$, in terms of $\dim(M)$, $\dim(G)$, k, and $\|R\|_\infty$.

1. Introduction and statement of a result. Bourguignon and Lawson [**B.L**] obtained the second variation formula of Yang-Mills functional

$$\mathcal{YM}(\nabla) = \frac{1}{2}\int_M \|R^\nabla\|^2,$$

introduced the notions of index $i(\nabla)$ and nullity $n(\nabla)$ of a Yang-Mills connection ∇, and showed that there are no Yang-Mills connections ∇ on S^n, $n \geq 5$, with the condition $i(\nabla) = 0$ (cf. [**B.L**, Theorem 7.11]). Therefore it would be interesting to give estimates of the index $i(\nabla)$ and nullity $n(\nabla)$ of Yang-Mills connections ∇.

Following [**B.L**], we prepare some notations.

Let G be a compact Lie group, and P a principal G-bundle over a compact Riemannian n-manifold M. For a faithful orthogonal representation ρ; $G \to O_N$, we consider a G-vector bundle $E = P \times_\rho \mathbf{R}^N$ associated to P by ρ. Each connection on P corresponds to a G-connection ∇ on E (cf. [**B.L**]). We denote by $\mathcal{C}_E$ the totality of G-connections on E. To each G-connection ∇ on E, the curvature R^∇, given by

$$R^\nabla_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

for tangent vectors X, Y on M, is a two form on M with values in the bundle $\mathfrak{so}_E$ whose fiber $\mathfrak{so}_{E,x}$, $x \in M$, consists of skew symmetric endomorphisms of the fiber E_x of E. The norm of R^∇ at each point x is given by

$$\|R^\nabla\|_x^2 = \sum_{i<j} \|R^\nabla_{e_i,e_j}\|^2,$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of the tangent space T_xM and the inner product of the fiber $\mathfrak{so}_{E,x}$ is given by

$$(A, B) = \tfrac{1}{2}\,\mathrm{trace}({}^tA \circ B),$$

for $A, B \in \mathfrak{so}_{E,x}$. The Yang-Mills functional $\mathcal{YM}$ on $\mathcal{C}_M$ is defined as

$$\mathcal{YM}(\nabla) = \frac{1}{2}\int_M \|R^\nabla\|^2$$

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for each $\nabla \in \mathcal{C}_E$. Moreover we define, for the sake of later use, the *supremum norm* $\|R^\nabla\|_\infty$ of R^∇ by

$$\|R^\nabla\|_\infty = \sup_{x\in M} \|R^\nabla\|_x. \tag{1.1}$$

There exists a subbundle $\mathfrak{g}_E$ of $\mathfrak{so}_E$ corresponding to a bundle $\mathfrak{g}_P := P \times_{\mathrm{Ad}} \mathfrak{g}$ through ρ. Let $\Omega^p(\mathfrak{g}_E)$, $0 \le p \le n$, be the space of $\mathfrak{g}_E$ valued p-forms on M. We get (cf. [**B.L**, p. 194]) an exterior differential d^∇; $\Omega^p(\mathfrak{g}_E) \to \Omega^{p+1}(\mathfrak{g}_E)$ and the adjoint operator δ^∇; $\Omega^p(\mathfrak{g}_E) \to \Omega^{p-1}(\mathfrak{g}_E)$ corresponding to $\nabla \in \mathcal{C}_E$. It is known (cf. [**B.L**]) that $B = \nabla - \nabla'$ belongs to $\Omega^1(\mathfrak{g}_E)$ for each ∇, $\nabla' \in \mathcal{C}_E$. An element ∇ in $\mathcal{C}_E$ is called a *Yang-Mills connection* if

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{YM}(\nabla^t) = 0,$$

for every one-parameter family of connections $\nabla^t \in \mathcal{C}_E$ with $\nabla^0 = \nabla$ and $d/dt|_{t=0}\nabla^t = B \in \Omega^1(\mathfrak{g}_E)$. The second variational formula is stated as follows.

THEOREM 1.1 (CF. [**B.L**, THEOREM 6.8]). *For a Yang-Mills connection $\nabla = \nabla^0 \in \mathcal{C}_E$ and $B = d/dt|_{t=0}\nabla^t \in \Omega^1(\mathfrak{g}_E)$ with $\delta^\nabla B = 0$, we have*

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathcal{YM}(\nabla^t) = \int_M (\mathcal{S}^\nabla(B), B),$$

where $\mathcal{S}^\nabla$ is a differential operator on $\Omega^1(\mathfrak{g}_E)$ given by

$$\mathcal{S}^\nabla(B) := \nabla^*\nabla B + B \circ \mathrm{Ric} + 2\mathcal{K}^\nabla(B), \qquad B \in \Omega^1(\mathfrak{g}_E).$$

Here $\nabla^*\nabla$ is the rough Laplacian on $\Omega^p(\mathfrak{g}_E)$ defined by

$$\nabla^*\nabla\varphi = -\sum_{j=1}^n \{\nabla_{e_j}\nabla_{e_j}\varphi - \nabla_{D_{e_j}e_j}\varphi\},$$

$\varphi \in \Omega^p(\mathfrak{g}_E)$, where D is the Levi-Civita connection of (M,g). The operator Ric; *$T_xM \to T_xM$ is the Ricci transform defined by*

$$\mathrm{Ric}(X) = \sum_{j=1}^n R_{X,e_j}e_j,$$

where R is the curvature tensor of (M,g). And $B\circ\mathrm{Ric}$ is defined by $(B\circ\mathrm{Ric})(X) := B(\mathrm{Ric}(X))$, $X \in T_xM$. The operator $\mathcal{K}^\nabla$; $\Omega^1(\mathfrak{g}_E) \to \Omega^1(\mathfrak{g}_E)$ is defined by

$$\mathcal{K}^\nabla(B)(X) = \sum_{j=1}^n [R^\nabla_{e_j,X}, B(e_j)].$$

Since $\mathcal{S}^\nabla$ is an elliptic, selfadjoint operator on $\Omega^1(\mathfrak{g}_E)$, its restriction to $T^0 := \mathrm{Ker}(\delta^\nabla) \subset \Omega^1(\mathfrak{g}_E)$ has eigenvalues $\lambda_1 < \lambda_2 < \cdots$ with associated finite-dimensional eigenspaces $E_{\lambda_1}, E_{\lambda_2}, \ldots$.

DEFINITION 1.2 (CF. [**B.L**, 6.10]). The index of a Yang-Mills connection ∇ is the dimension $i(\nabla) := \dim(\bigoplus_{\lambda<0} E_\lambda)$, and the nullity of ∇ is the dimension $n(\nabla) := \dim(E_0)$.

Now we can state our main theorem.

THEOREM 1.3. *Assume that* (M, g) *is a compact Riemannian manifold whose Ricci tensor satisfies* $\mathrm{Ric} \geq kg$ *with a positive constant* k. *Then the index* $i(\nabla)$ *and the nullity* $n(\nabla)$ *of a Yang-Mills connection* ∇ *over* M *can be estimated as follows:*

(I) *If* $k - 2\sqrt{2}n\|R^\nabla\|_\infty > 0$, *then* $i(\nabla) = n(\nabla) = 0$.

(II) *If* $k - 2\sqrt{2}n\|R^\nabla\|_\infty = 0$, *then* $i(\nabla) = 0$ *and* $n(\nabla) \leq \dim(G)$.

(III) *In case* $k - 2\sqrt{2}n\|R^\nabla\|_\infty < 0$:

(i) *If* $n \geq 3$, *we have*

$$i(\nabla) + n(\nabla) \leq \dim(G)(1 + \tfrac{1}{A})^A\{1 + (n-1)! n^{n-1} A(1+A)^{n-1}\},$$

where

$$A := 2\sqrt{2}k^{-1}\|R^\nabla\|_\infty - n^{-1} = (nk)^{-1}(2\sqrt{2}n\|R^\nabla\|_\infty - k).$$

(ii) *If* $n = 2$, *we have*

$$i(\nabla) + n(\nabla) \leq \dim(G)(1 + \tfrac{1}{B})^B\{1 + 4B^2\},$$

where

$$B := 4\sqrt{2}k^{-1}\|R^\nabla\|_\infty - 1 = k^{-1}(4\sqrt{2}\|R^\nabla\|_\infty - k).$$

REMARK 1.4. The function $(1 + 1/x)^x$ in x satisfies

$$\lim_{x\to 0}(1 + \tfrac{1}{x})^x = 1 \quad \text{and} \quad (1 + \tfrac{1}{x})^x < e \quad \text{for all } x > 0.$$

2. Proof of Theorem 1.3. Assume that the Ricci tensor of (M, g) satisfies $\mathrm{Ric} \geq kg$ with $k > 0$. Put $\mathcal{P}^\nabla(B) := B \circ \mathrm{Ric} + 2\mathcal{K}^\nabla(B)$, $B \in \Omega^1(\mathfrak{g}_E)$. Then we obtain

LEMMA 2.1. *We have*

$$(\mathcal{P}^\nabla(B), B)_x \geq (k - 2\sqrt{2}n\|R^\nabla\|_\infty)(B, B)_x,$$

for all $B \in \Omega^1(\mathfrak{g}_E)$. *Here* $(\cdot,\cdot)_x$ *is the pointwise inner product on* $\Omega^1(\mathfrak{g}_E)$ *as in* §1.

PROOF. Since $(B \circ \mathrm{Ric}, B)_x \geq k(B, B)_x$, $B \in \Omega^1(\mathfrak{g}_E)$, we have only to show, for each $B \in \Omega^1(\mathfrak{g}_E)$,

$$|(\mathcal{K}^\nabla(B), B)_x| \leq \sqrt{2}n\|R^\nabla\|_\infty(B, B)_x.$$

Since $(\mathcal{K}^\nabla(B), B)_x$ coincides, by definition, with

$$\sum_{j=1}^n (\mathcal{K}^\nabla(B)(e_j), B(e_j))_x = \sum_{j,k=1}^n ([R^\nabla_{e_k,e_j}, B(e_k)],\ B(e_j))_x,$$

we have, using the Cauchy-Schwarz inequality,

$$\begin{aligned}|(\mathcal{K}^\nabla(B), B)_x| &\leq \sum_{j,k=1}^n \|[R^\nabla_{e_k,e_j}, B(e_k)]\|_x \|B(e_j)\|_x \\ &\leq \sqrt{2}\sum_{j,k=1}^n \|R^\nabla_{e_k,e_j}\|_x \|B(e_k)\|_x \|B(e_j)\|_x,\end{aligned}$$

due to the inequality in [**B.L**, Lemma 2.30]. Therefore, by definition of $\|R^\nabla\|_\infty$, the right-hand side is smaller than or equal to

$$\sqrt{2}\|R^\nabla\|_\infty \sum_{k,j=1}^{n} \|B(e_k)\|_x \|B(e_j)\|_x = \sqrt{2}\|R^\nabla\|_\infty \left(\sum_{j=1}^{n} \|B(e_j)\|_x\right)^2$$
$$\leq \sqrt{2}n\|R^\nabla\|_\infty \sum_{j=1}^{n} \|B(e_j)\|_x^2 = \sqrt{2}n\|R^\nabla\|_\infty \|B\|_x^2,$$

by the Cauchy-Schwarz inequality. Q.E.D.

Now we give an estimation of $i(\nabla) + n(\nabla)$ of a Yang-Mills connection ∇ using the heat equation method due to Bérard-Gallot [**B.G**]. We denote the eigenvalues of $S^\nabla|_{T^0}$, S^∇ and $\nabla^*\nabla$ counted with their multiplicities by $\tilde{\tilde{\lambda}}_i$, $\tilde{\lambda}_i$, and $\bar{\lambda}_i$, respectively. Then by the mini-max principle, we get, for each i,

$$\tilde{\tilde{\lambda}}_i \geq \tilde{\lambda}_i \geq \bar{\lambda}_i + k - 2\sqrt{2}n\|R^\nabla\|_\infty.$$

Therefore for each $t > 0$,

$$i(\nabla) + n(\nabla) \leq \sum_{i=1}^{\infty} e^{-t\tilde{\tilde{\lambda}}_i} \leq \sum_{i=1}^{\infty} e^{-t\tilde{\lambda}_i} \tag{2.1}$$
$$\leq e^{-t(k-2\sqrt{2}n\|R^\nabla\|_\infty)} \sum_{i=1}^{\infty} e^{-t\bar{\lambda}_i}.$$

Applying Kato's type inequality due to Hess, Schrader and Uhlenbrock in [**H.S.U**] to our case of the rough Laplacian $\nabla^*\nabla$ on $\Omega^1(\mathfrak{g}_E)$, we obtain, for each $t > 0$,

$$i(\nabla) + n(\nabla) \leq \dim(G) e^{-t(k-2\sqrt{2}n\|R^\nabla\|_\infty)} Z_M(t). \tag{2.2}$$

Here $Z_M(t)$ is the trace of the heat kernel of $\partial/\partial t + \Delta_M$ of smooth functions on M, and Δ_M is the (positive) Laplacian of (M, g). Moreover applying the inequality in [**B.G**, Corollary 2.2] to our Riemannian manifold (M, g) with $\mathrm{Ric} \geq kg$, $k > 0$, we get

$$Z_M(t) \leq Z_{S^n}(kt), \qquad t > 0,$$

where $Z_{S^n}(t)$ is the trace of the heat kernel of the standard unit sphere (S^n, can). Therefore we obtain

$$i(\nabla) + n(\nabla) \leq \dim(G) \mathrm{Inf}\{e^{-t(k-2\sqrt{2}n\|R^\nabla\|_\infty)} Z_{S^n}(kt); t > 0\}. \tag{2.3}$$

It is known (cf. [**B.G.M**]) that, if $n \geq 2$,

$$Z_{S^n}(t) = \sum_{k=0}^{\infty} m_k e^{-tk(k+n-1)},$$

where

$$m_k = \frac{(n+k-2)!}{k!(n-1)!}(n+2k-1), \qquad k = 0, 1, 2, \ldots.$$

Then $Z_{S^n}(t)$ is estimated as follows:

(i) If $n \geq 3$,

$$Z_{S^n}(t) \leq 1 + \sum_{k=1}^{\infty}(nk)^{n-1}e^{-tnk}$$
$$\leq 1 + (n-1)!n^{n-1}e^{-tn}(1-e^{-tn})^{-n}.$$

(ii) If $n = 2$,

$$Z_{S^2}(t) \leq 1 + 2\sum_{k=2}^{\infty} ke^{-tk}$$
$$\leq 1 + 2e^{-2t}(2-e^{-t})(1-e^{-t})^{-2}.$$

Using these inequalities and putting $e^t = 1 + 1/A$ in (2.3) in case of (i), or $e^t = 1 + 1/B$ in (2.3) in case of (ii), respectively, we obtain the desired estimates of $i(\nabla) + n(\nabla)$ in case of (III). In case of (I) or (II), using the inequalities (2.1), (2.2) and the fact that $\lim_{t\to\infty} Z_M(t) = 1$, we obtain the desired results. Q.E.D.

REMARK 2.2. The analogous estimates of indices and nullities for harmonic maps are obtained in [**U**].

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