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A TOPOLOGICAL INTERPRETATION FOR THE BIAS INVARIANT

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ABSTRACT. The bias invariant has been used to distinguish between the homotopy types of 2-complexes. In this note we show that two finite, connected 2-complexes X and Y with isomorphic fundamental groups and the same Euler characteristic have the same bias invariant if and only if there is a map $f\colon X \to Y$ which is a homology equivalence ($\pi_1 f$ and $H_2 f$ are isomorphisms).

Let G be a group and let d be an integer $\geqslant 2$. A $[G, d]$-complex X is a connected CW-complex of dimension $\leqslant d$ whose fundamental group is isomorphic to G and whose universal covering $\tilde{X}$ is $(d-1)$-connected. For any two $[G, d]$-complexes X, X' there is a comparison invariant called *bias*. This invariant has been used in [**DS**, **SD**, **D**, **WB**, $\mathbf{D_2}$, **Sc**, **M**, and **S**] to detect when two such complexes are not homotopy equivalent.

Given any integer $d \geqslant 2$, any two $[G, d]$-complexes X, X', and any $\alpha \in \operatorname{Hom}(\pi_1 X, \pi_2 X')$, there exists a continuous function $f\colon X \to X'$ such that $f_{1\#} = \alpha$. If X, X' are *finite* $[G, d]$-complexes having the same Euler characteristic, does there always exist a map $f\colon X \to X'$ such that $\pi_1 f$ and $H_* f$ are isomorphisms? We call such a map a *homology equivalence* and such spaces *homology equivalent*. The purpose of this note is to show that the *bias* detects *exactly* whether or not two $[G, d]$-complexes are homology equivalent. This has also been independently observed by Wolfgang Metzler.

To fix notation we observe that, if X is a finite $[G, d]$-complex, then the augmented cellular chain complex $C_*\tilde{X} \to \mathbf{Z}$ forms a *partial*, finitely generated, free resolution of length d, called a $[\pi_1 X, d]$-*resolution*. For two such complexes X, X' a *comparison chain map* is any chain map $k\colon C_* X \to C_* X'$ which is the identity on $H_0 C_* \approx \mathbf{Z}$. Any two such chain maps are chain homotopic.

If M, F are free abelian groups and $i\colon M \rightarrowtail F$ is the inclusion, then the *Fox ideal* $\mathscr{F}(i)$ is ideal in $\mathbf{Z}$ generated by all coordinates of the elements $i(x)$ for all $x \in M$. If one diagonalizes the matrix for i, call it $\operatorname{diag}(i)$; then it is easy to see that [**D**, 3.8]

$$\mathscr{F}(i) = \begin{cases} \mathbf{Z} & \text{if } \operatorname{diag}(i) \text{ has at least one 1 or } F/M \text{ has no torsion,} \\ (\text{g.c.d.}\{\text{torsion coefficients of } F/M\}) & \\ & \text{if } F/M \text{ has torsion } and \operatorname{diag}(i) \text{ has no 1's.} \end{cases}$$

We write $\mathscr{F}(i) = (m)$, and call m the *modulus* of $i\colon M \to F$.

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For each $[G, d]$-complex X, we define a free abelian group $\beta\pi_d X \subseteq \mathbf{Z} \otimes_G C_d\tilde{X}$ to be the image of $1 \otimes j\colon \mathbf{Z} \otimes_G \pi_d X \to \mathbf{Z} \otimes_G C_d X$ induced by the natural inclusion $j\colon \pi_d X \to C_d\tilde{X}$.

Fix a $[G, d]$-complex X and let $(m) = \mathscr{F}(i)$ be the Fox ideal of the inclusion $i\colon \beta\pi_d X \subseteq \mathbf{Z} \otimes_G C_d\tilde{X}$. Notice that the functor $\rho_m = \mathbf{Z}_m \otimes -$ has $\rho_m(i) = 0$. Such a functor is called a "killing functor" in **[D**, §7].

For each $\alpha \in \operatorname{Aut}(\pi_1 X)$, define $b(\alpha) \in \mathbf{Z}_m^*/\pm 1$ as follows. Consider the cellular chain complex $C_*\tilde{X} \to \mathbf{Z}$ of the universal cover $\tilde{X}$ of X as a $[G, d]$-resolution by identifying $\pi_1 X = G$. For each α, and each G-module M, let ${}_\alpha M$ denote the G-module with action $g * x = \alpha(g) \cdot x$ for $g \in G$ and $x \in M$. Let $i'\colon \pi_d X \to C_d\tilde{X}$ be the inclusion. Let $k(X,1;\, X,\alpha)$ be any map $k(X,1;\, X,\alpha)\colon \pi_d X \to {}_\alpha\pi_d X$ arising from the comparison of $C_*\tilde{X} \twoheadrightarrow \mathbf{Z}$ to ${}_\alpha C_*\tilde{X} \twoheadrightarrow \mathbf{Z}$. Because the diagram

$$\begin{array}{cccc} & \pi_d X & \overset{i'}{\rightarrowtail} & C_d\tilde{X} \\ k(X,1;\, X,\alpha) & \downarrow & & \downarrow k_d \\ & {}_\alpha\pi_d X & \overset{\alpha^{i'}}{\rightarrowtail} & {}_\alpha C_d\tilde{X} \end{array}$$

commutes, we may define $\beta(k(X,1;\, X,\alpha))\colon \beta\pi_d X \to \beta_\alpha\pi_d X$ (functorially). Using the right exact functor $\mathbf{Z} \otimes_G -$, then the bias $b(X,1;\, X,\alpha)$ is the class of $\det[\beta k(X,1;\, X,\alpha)\colon \beta\pi_d X \to \beta_\alpha\pi_d X]$ modulo m.

Similarly, if we consider ${}_\beta C_* X' \twoheadrightarrow \mathbf{Z}$, $\beta \in \operatorname{Iso}(G, \pi_1 X')$, then we may define

$$k(X,\alpha;\, X',\beta)\colon {}_\alpha\pi_d X \to {}_\beta\pi_d X' \qquad (\alpha \in \operatorname{Aut} G)$$

as the comparison k-invariant between ${}_\alpha C_* X \twoheadrightarrow \mathbf{Z}$ and ${}_\beta C_* X' \twoheadrightarrow \mathbf{Z}$. Then $b(X,\alpha;\, X',\beta) \in \mathbf{Z}_m^*/\pm 1$ is given by the class of $\det[\beta k(X,\alpha;\, X',\beta)]$ (modulo m).

The definitions above are well defined because $\mathscr{F}_m = \mathbf{Z}_m \otimes (\mathbf{Z} \otimes_G \cdot)$ is a killing functor for $i'\colon \pi_d X \rightarrowtail C_d\tilde{X}$ **[D**, §7]. Clearly, also the Fox ideal of $\beta\pi_d X \rightarrowtail \mathscr{F}C_d\tilde{X}$ is the same as $\beta_\alpha\pi_d X \rightarrowtail \mathscr{F}_\alpha C_d\tilde{X}$ ($\alpha \in \operatorname{Aut} G$). Arguments from §7 (or §4) of **[D]** show that $b(X,\alpha;\, X',\beta)$ is independent of the choice of $k(X,\alpha;\, X',\beta)$. Furthermore, it follows from 3.8 of **[D]** that for any $[G, d]$-complex X', the Fox ideal of $i\colon \beta\pi_d X' \rightarrowtail \mathscr{F}C_d\tilde{X}'$ is (m), where m is either 1 or the g.c.d. of the torsion coefficients of $H_d(G)$.

LEMMA 1. *Let* $\alpha \in \operatorname{Aut} G$ *and* $\gamma \in \operatorname{Iso}(G, \pi_1 X')$. *Then*
(a) $b(X,\alpha;\, X',\gamma) = b(X,1;\, X',\gamma\alpha^{-1})$ *and*
(b) $b(X,1;\, X',\gamma\alpha) = b(X,1;\, X',\gamma) \cdot b(X,1;\, X,\alpha)$.

PROOF. (a) If we compare $C_*\tilde{X} \twoheadrightarrow \mathbf{Z}$ and ${}_{\gamma\alpha^{-1}}C_*\tilde{X}' \twoheadrightarrow \mathbf{Z}$, we obtain $k(X,1;\, X',\gamma\alpha^{-1})$. Then, applying α to both sequences yields ${}_\alpha k(X,1;\, X',\gamma\alpha^{-1}) = k(X,\alpha;\, X',\gamma)$. Hence

$$\beta k(X,\alpha;\, X',\gamma) = \left(\beta_\alpha k(X,1;\, X',\gamma\alpha^{-1})\right) = \beta k(X,1;\, X',\gamma\alpha^{-1}),$$

because $\beta_\alpha\pi_d X = \beta\pi_d X$, as trivial G-modules. Hence, $b(X,\alpha;\, X',\gamma) = b(X,1;\, X',\gamma\alpha^{-1})$.

(b) By comparing $C_*\tilde{X} \to \mathbf{Z}$ to ${}_\alpha C_*\tilde{X} \to \mathbf{Z}$ and then ${}_\alpha C_*\tilde{X} \twoheadrightarrow \mathbf{Z}$ to ${}_{\gamma\alpha}C_*\tilde{X}' \twoheadrightarrow \mathbf{Z}$ we obtain

$$k(X,1;\, X',\gamma\alpha) = k(X,\alpha;\, X',\gamma\alpha)k(X,1;\, X,\alpha)$$

and hence

$$b(X,1;\, X',\gamma\alpha) = b(X,1;\, X',\gamma)b(X,1;\, X,\alpha)$$

by (a). $\square$

If we let $D(G,d)$ denote the image of b: $\operatorname{Aut} G \to \mathbf{Z}_m^*/\pm 1$ given by the *homomorphism* $\alpha \to b(X,1;\, X,\alpha)$, we see that the *comparison bias*

$$b(X,X') = b(X,1;\, X',\gamma) \cdot D(G,d) \in \mathbf{Z}_m^*/\pm D(G,d)$$

is independent of the choice of the "polarization" $\gamma \in \operatorname{Iso}(G,\pi_1 X')$. This follows from Lemma 1(b).

THEOREM 2. *For each pair of finite* $[G,d]$*-complexes* X, X' *with the same Euler characteristic,* X *is homology equivalent to* X' *iff* $b(X,X') = 1$ *in* $\mathbf{Z}_m^*/\pm D(G,d)$.

PROOF. First, we identify $\beta\pi_d X$ as the group $\Sigma_d X$ of spherical elements in $H_d X$; i.e. $\beta\pi_d X$ is the image of the Hurewicz map h_d: $\pi_d X \to H_d X$. Notice that any f: $X \to X'$ which induces an isomorphism α on π_1 automatically induces an isomorphism $H_s X \to H_s X'$ for $s < d$. This follows because in these dimensions $H_s X \cong H_s G$. Further, f induces a commutative diagram

$$\text{(3)}\qquad \begin{array}{ccccc} \beta\pi_d X & \overset{i}{\rightarrowtail} & H_d X & \twoheadrightarrow & H_d G \\ \downarrow \bar{i} & \searrow\ \swarrow & \downarrow f_{d*} & & \cong\ \downarrow H_d(\pi_1 f) \\ & C_2 X & & & \\ \beta\pi_d X' & \overset{i'}{\rightarrowtail} & H_d X' & \twoheadrightarrow & H_d G. \end{array}$$

Note that $H_d X \cong H_d X'$ as finitely generated free abelian groups, because X, X' have the same Euler characteristic. The only question is, can we adjust f so that $\pi_1 f$ is still an isomorphism and f_{d*} is an isomorphism.

If $\Sigma_d X = 0$, then $\beta\pi_d X' = 0$ also, and f_{d*} is an isomorphism for any f with $\pi_1 f$ an isomorphism, by the above reasoning.

Now, let us suppose that $\Sigma_d X \neq 0$ and that $b(X,X') = 1$ inside $\mathbf{Z}_m^*/\pm D(G,d)$. This means there are homomorphisms $\alpha \in \operatorname{Aut} G$, $\beta \in \operatorname{Iso}(G,\pi_1 X')$ so that

$$b(X,1;\, X',\beta) \cdot b(X,1;\, X,\alpha) = b(X,1;\, X',\beta\alpha) \equiv \pm 1 \bmod m.$$

Let ${}_{\beta\alpha}\pi_d X'$ be denoted by N. *We claim that there exists a map* $\bar{\gamma}$: $C_d X \to \beta N$ *so that* $\beta k(X,1;\, X',\beta\alpha) + \bar{i}\bar{\gamma}$ *is an isomorphism from* $\beta\pi_d X \to \beta N$:

$$\begin{array}{ccc} \beta\pi_d X & \overset{\bar{i}}{\rightarrow} & C_d X \\ \beta k \downarrow & \overset{\bar{\gamma}}{\swarrow} & \downarrow \\ \beta N & \rightarrowtail & C_d X'. \end{array}$$

To see this, we choose bases for the free abelian groups in (4) so that i, i' are diagonal:

$$
\begin{tikzcd}
\beta\pi_d X \arrow[r, tail, "i"] \arrow[d, "\beta k"'] & H_d X \arrow[r] \arrow[d, "f_{d*}"] \arrow[dl, dashed, "\gamma"'] & H_d G \arrow[d, "H_d(\beta\alpha)\ \approx"] \\
\beta N \arrow[r, tail, "i'"] & H_d X' \arrow[r, two heads] & H_d\pi_1 X' \approx H_d G.
\end{tikzcd} \tag{4}
$$

We may then canonically identify $H_dG = H_d\pi_1X'$ and view $H_d(\beta\alpha)$ as a member of $\operatorname{Aut} H_dG$, a finitely generated abelian group. By Lemma 5.1 of [**D**], $\det \beta k = \pm\det f_{d*} \equiv \pm 1 \bmod (m)$. Hence, $\det H_d(\beta\alpha) \equiv \pm 1 \bmod (m)$. Now, if $m =$ g.c.d. torsion coefficients of H_dG, we may use the argument of Theorem 6.9 of [**D**] (using the full strength of Appendix B) to find $\gamma\colon H_dX \to \beta N$ so that $\det(f_{d*} + i'\gamma) = \pm 1$. Because H_dX is a direct summand of C_dX, let a projection $\rho\colon C_dX \twoheadrightarrow H_dX$ be given. Then define $\bar\gamma = \gamma\rho$.

If $m = 1$ it is because either rank $H_dX >$ the minimal number of generators of H_dG or $H_d(G)$ is free abelian. In either case, one may choose γ so that $\det(f_{d*} + i'\gamma)$ is *any* preassigned number; so choose γ so that the determinant is one. Then choose $\bar\gamma$ as above. This finishes the proof of the claim.

Let $\mathscr{F}$ denote the functor $\mathbf{Z} \otimes_G -$. We observe that $q\mathscr{F}\colon \operatorname{Hom}_{\mathbf{Z}G}(C_d\tilde X, N) \to \operatorname{Hom}_{\mathbf{Z}}(C_dX, \beta N)$ is surjective, where $q\colon \mathscr{F}N \twoheadrightarrow \beta N$ is essentially $\mathscr{F}\{i'\colon N \to C_dX'\}$. So choose $\gamma\colon C_d\tilde X \to N$ so that $q\mathscr{F}\gamma = \bar\gamma$. Now let $k'(X, 1; X', \beta\alpha) = k(X, 1; X', \beta\alpha) + \gamma j$, where $j\colon \pi_2X \to C_d\tilde X$ is the inclusion. But this $k'\colon \pi_dX \to {}_{\beta\alpha}\pi_dX'$ is realizable as a π_df', $f'\colon X \to X'$, with $\pi_1 f' = \beta\alpha$. (See [**DS**, p. 41], for a proof of the realizability of any chain map $C_*X \to C_*X'$.) Thus $b(X, X') = 1$ in $\mathbf{Z}_m^*/\pm D(G, d)$ implies that X and X' are homology equivalent.

The converse is easy. If $f\colon X \to X'$ is a map which is a homology equivalence, then f induces

$$
\begin{array}{ccccc}
\beta\pi_d X & \to & H_dX & \to & H_dG \\
\downarrow \cong \beta k(X,1;X',\alpha) & & \downarrow \cong & & \downarrow \cong \\
\beta_\alpha\pi_d X' & \to & H_dX' & \to & H_dG
\end{array}
$$

where $\alpha = \pi_1 f$. Hence $\det \beta k(X, 1; X', \alpha) = \pm 1$; so $b(X, 1; X'; \alpha) = 1$ inside $\mathbf{Z}_m^*/\pm D(G, d)$ [**D**, 5.1]. □

COROLLARY 5. *If X, X' are finite $[G, 2]$-complexes with the same Euler characteristic and $m = 1$, then X is homology equivalent to X'.* □

This always is satisfied if $\chi(X) > \chi_{\min}(G, d)$ ($=$ the minimal Euler characteristic for all finite $[G, d]$-complexes).

The amazing fact is that, in many cases, homology equivalence implies homotopy equivalence [**D$_2$**, **DS**, **WB**]. However, this is not the case with Dunwoody's example [**Du**] where G is the trefoil group and $\chi(X) = \chi_{\min}(G, 2) + 1$, with X a $[G, 2]$-complex. In this case $m = 1$ and $H_2G = 0$. If Y is the realization of the presentation $\{a, b\colon a^3 = b^2\}$ of G, then X and $X' = Y \vee S^2$ are homology equivalent, but *not* homotopy equivalent.

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