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A NOTE ON THE AUTOMORPHISM GROUPS OF LIE MODULE TRIPLE SYSTEMS

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ABSTRACT. Using representation theory and well-known facts about automorphism groups of reductive Lie algebras, the automorphism group of a basic Lie module triple system $(M, \{\ ,\ ,\ \}, \mathcal{L}, b, \phi)$ over an algebraically closed field of characteristic zero is related to the automorphism group of the Lie algebra $\mathcal{L}$, the automorphism group of the standard embedding $\mathcal{S}$ of $(M, \{\ ,\ ,\ \})$, and the automorphism group of the split null extension $\overline{\mathcal{S}}$ of $\mathcal{L}$ by M.

1. Introduction. The problem considered in this paper is that of determining the automorphism group $\mathrm{Aut}(M, \{\ ,\ ,\ \})$ of a basic Lie module triple system (LMTS) $(M, \{\ ,\ ,\ \}, \mathcal{L}, b, \phi)$ over an algebraically closed field k of characteristic zero. We will do this by relating $\mathrm{Aut}(M, \{\ ,\ ,\ \})$ to the automorphism group $\mathrm{Aut}\,\mathcal{L}$ of the reductive Lie algebra $\mathcal{L}$ and using well-known facts about automorphism groups of reductive Lie algebras (see §2 for definitions).

The relationship between $\mathrm{Aut}(M, \{\ ,\ ,\ \})$ and $\mathrm{Aut}\,\mathcal{L}$ is seen by defining two algebra structures $\mathcal{S}$, the standard embedding of $(M, \{\ ,\ ,\ \})$, and $\overline{\mathcal{S}}$, the split null extension of $\mathcal{L}$ by M, on $M \oplus \mathcal{L}$. The groups $\mathrm{Aut}(\mathcal{S}, \sigma)$ and $\mathrm{Aut}(\overline{\mathcal{S}}, \sigma)$ of automorphisms of $\mathcal{S}$ and $\overline{\mathcal{S}}$, respectively, stabilizing M and $\mathcal{L}$ have natural restriction maps $r\colon \mathrm{Aut}(\mathcal{S}, \sigma) \to \mathrm{Aut}\,\mathcal{L}$ and $\overline{r}\colon \mathrm{Aut}(\overline{\mathcal{S}}, \sigma) \to \mathrm{Aut}\,\mathcal{L}$ which are homomorphisms. Since $\mathcal{S}$ is generated by $(M, \{\ ,\ ,\ \})$ it is not surprising that $\mathrm{Aut}(M, \{\ ,\ ,\ \}) \cong \mathrm{Aut}(\mathcal{S}, \sigma)$ and so $\mathrm{Aut}(M, \{\ ,\ ,\ \})$ will be determined once the kernel and image of r are known. However, since $\overline{\mathcal{S}}$ only involves the $\mathcal{L}$-module structure of M, the kernel and image of $\overline{r}$ are much easier to compute using standard representation theory facts (Theorems 3.1 and 3.2 and Lemma 4.1). Since $\mathrm{Aut}(\mathcal{S}, \sigma) \subseteq \mathrm{Aut}(\overline{\mathcal{S}}, \sigma)$, using Lemma 3.5, Corollary 3.6, and Lemma 4.2 it is possible to determine the kernel and image of r from those of $\overline{r}$ (Theorems 3.7 and 3.9 and Lemma 4.3). The surprising conclusion is that if ϕ is symmetric or symplectic, the images of r and $\overline{r}$ are the same (Corollary 3.8), $\mathrm{Aut}(\mathcal{S}, \sigma)$ is a normal subgroup of $\mathrm{Aut}(\overline{\mathcal{S}}, \sigma)$, and $\mathrm{Aut}(\overline{\mathcal{S}}, \sigma)/\mathrm{Aut}(\mathcal{S}, \sigma) \cong k^*$ (Theorem 4.4).

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2. Basic definitions. Recall from [**3**] that a Lie module triple system (abbreviated LMTS) is formed from a finite-dimensional Lie algebra $\mathcal{L}$ having a nondegenerate symmetric associative bilinear form b and a finite-dimensional faithful $\mathcal{L}$-module M having a nondegenerate $\mathcal{L}$-invariant bilinear form ϕ, that is

$$\phi(xl, y) = -\phi(x, yl) \tag{2.1}$$

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for all $x, y \in M$, $l \in \mathcal{L}$. A triple product $\{\ ,\ ,\ \}$ on M is defined by setting $\{xyz\} := xR(y,z)$ where $R\colon M \times M \to \mathcal{L}$ is given by

$$b(l, R(y,z)) = \phi(zl, y) \tag{2.2}$$

for all $y, z \in M$, $l \in \mathcal{L}$. The *Lie module triple system* $(M, \{\ ,\ ,\ \})$ is denoted $(M, \{\ ,\ ,\ \}, \mathcal{L}, b, \phi)$ when the ingredients need to be specified.

We will always assume that $(M, \{\ ,\ ,\ \}, \mathcal{L}, b, \phi)$ is a simple LMTS such that M is completely reducible as an $\mathcal{L}$-module. Hence $\mathcal{L}$ is reductive (Theorem 10, p. 81 of [**4**]), i.e. $\mathcal{L} = [\mathcal{L}, \mathcal{L}] \oplus C$ with $[\mathcal{L}, \mathcal{L}]$ semisimple and C a central ideal of $\mathcal{L}$ whose elements act semisimply on M. Λ will denote the set of lowest weights, without multiplicities, of M as an $\mathcal{L}$-module. We will usually assume that $(M, \{\ ,\ ,\ \}, \mathcal{L}, b, \phi)$ is one of the following *basic* types of LMTS's:

Type (I). M is an irreducible self-dual $\mathcal{L}$-module of lowest weight λ, $\mathcal{L}$ is semisimple, and ϕ is symmetric or symplectic.

Type (II). $M = M_1 \oplus M_2$ with M_1 and M_2 isomorphic irreducible self-dual $\mathcal{L}$-modules of lowest weight λ, $\mathcal{L}$ is semisimple, ϕ is symmetric or symplectic, and ϕ restricted to $M_i \times M_i$ is identically zero for $i = 1, 2$. $\mathrm{Der}(M, \{\ ,\ ,\ \}) = \mathcal{L} \oplus \mathrm{sl}(2)$ where $\mathrm{Der}(M, \{\ ,\ ,\ \})$ is the derivation algebra of $(M, \{\ ,\ ,\ \})$ and $\mathcal{L}, \mathrm{sl}(2)$ are ideals of $\mathrm{Der}(M, \{\ ,\ ,\ \})$.

Type (III). $M = M_1 \oplus M_2$ with M_1 and M_2 irreducible dual $\mathcal{L}$-submodules of lowest weight λ_1 and λ_2 respectively with $\lambda_1 \neq \lambda_2$. ϕ restricted to $M_i \times M_i$ is identically zero and there is a $z \in k^*$ with $\phi(z, y) = a\phi(y, z)$ for all $z \in M_1$, $y \in M_2$. $\mathcal{L}$ is semisimple or has a one-dimensional center spanned by c where $(x_1 + x_2)c := x_1 - x_2$ for $x_i \in M_i$.

Note that simple Lie triple systems are basic [**5**, **1**]. Other examples of basic LMTS's are given in [**1**, **2**] where it was shown that if $(M, \{\ ,\ ,\ \}, \mathcal{L}, b, \phi)$ is a LMTS for which M is a completely reducible $\mathcal{L}$-module and ϕ is symmetric or symplectic, then $(M, \{\ ,\ ,\ \})$ can be constructed from basic LMTS's and one- and two-dimensional abelian LMTS's. Note however that not all simple completely reducible LMTS's are basic (see [**1**] for examples of nonbasic simple LMTS's).

$\alpha \in \mathrm{GL}(M)$ is an *automorphism* of $(M, \{\ ,\ ,\ \})$ if $\{xyz\}\alpha = \{x\alpha, y\alpha, z\alpha\}$ and $\mathrm{Aut}(M, \{\ ,\ ,\ \})$ will denote the automorphism group of $(M, \{\ ,\ ,\ \}, \mathcal{L}, b, \phi)$. There are two nonassociative algebra structures naturally defined on $M \oplus \mathcal{L}$, the first being the split null extension $\overline{\mathcal{S}}$ of $\mathcal{L}$ by M, with multiplication defined by $(x+l)(y+l_1) := xl_1 - yl + [l, l_1]$, and the second being the *standard embedding* $\mathcal{S}$ of $(M, \{\ ,\ ,\ \})$ (see [**3**]), with multiplication defined by $(x+l)(y+l_1) := xl_1 - yl + [l, l_1] + R(x, y)$ for $x, y \in M$, $l, l_1 \in \mathcal{S}$. $\sigma \in \mathrm{GL}(M \oplus \mathcal{L})$ defined by $(x + l)\sigma := -x + l$ is an automorphism of both $\mathcal{S}$ and $\overline{\mathcal{S}}$ fixing $\mathcal{L}$. Let $\mathrm{Aut}(\mathcal{S}, \sigma)$ (respectively $\mathrm{Aut}(\overline{\mathcal{S}}, \sigma)$) be the automorphisms of $\mathcal{S}$ (respectively $\overline{\mathcal{S}}$) which commute with σ.

Clearly $\mathrm{Aut}(\mathcal{S}, \sigma)$ is a subgroup of$\mathrm{Aut}(\overline{\mathcal{S}}, \sigma)$ and $\psi\colon \mathrm{Aut}(M, \{\ ,\ ,\ \}) \to \mathrm{Aut}(\mathcal{S}, \sigma)$ defined by $(x + l)(\tau\psi) := x\tau + (\tau^{-1}l\tau)$ for $\tau \in \mathrm{Aut}(M, \{\ ,\ ,\ \})$ is an isomorphism. For computing $\mathrm{Aut}(\overline{\mathcal{S}}, \sigma)/\mathrm{Aut}(\mathcal{S}, \sigma)$ we will also need to consider the automorphism group $\mathrm{Aut}\,\mathcal{L}$ of $\mathcal{L}$ and the homomorphisms $r\colon \mathrm{Aut}(\mathcal{S}, \sigma) \to \mathrm{Aut}\,\mathcal{L}$ and $\overline{r}\colon \mathrm{Aut}(\overline{\mathcal{S}}, \sigma) \to \mathrm{Aut}\,\mathcal{L}$ given by restriction, i.e. $\tau r := \tau|_{\mathcal{L}}$ and $\eta\overline{r} := \eta|_{\mathcal{L}}$ for $\tau \in \mathrm{Aut}(\mathcal{S}, \sigma)$, $\eta \in \mathrm{Aut}(\overline{\mathcal{S}}, \sigma)$. The kernel of r (respectively $\overline{r}$) is denoted $\ker r$ (respectively $\ker \overline{r}$) and the image of r (respectively $\overline{r}$) by $\mathrm{im}\, r$ (respectively $\mathrm{im}\, \overline{r}$).

3. Images. Since any automorphism of $\mathcal{L}$ is the product of an inner automorphism and one that stabilizes a Borel subalgebra of $\mathcal{L}$ [**4**], we will consider these two types of automorphisms of $\mathcal{L}$ separately. $\operatorname{Inn}\operatorname{Aut}\mathcal{L}$ will be the group of inner automorphisms of $\mathcal{L}$. For $e \in \mathcal{L}$ nilpotent define $\exp(\operatorname{ad}_S e) \in \operatorname{Aut}(S,\sigma)$ by

$$\exp(\operatorname{ad}_S e) := \sum_{n=0}^{\infty} \frac{1}{n!}(\operatorname{ad}_S e)^n \quad \text{where } x(\operatorname{ad}_S e) := xe \text{ for all } x \in S.$$

THEOREM 3.1 (COROLLARY 5, p. 44 OF [**6**]). *Let* $\operatorname{Inn}\operatorname{Aut}(S,\sigma)$ *be the subgroup of* $\operatorname{Aut}(S,\sigma)$ *generated by* $\{\exp(ad_S e) | e \in \mathcal{L},\ e \text{ nilpotent}\}$. *Then* $\tau r \in \operatorname{Inn}\operatorname{Aut}\mathcal{L}$ *for* $\tau \in \operatorname{Inn}\operatorname{Aut}(S,\sigma)$, $r\colon \operatorname{Inn}\operatorname{Aut}(S,\sigma) \to \operatorname{Inn}\operatorname{Aut}\mathcal{L}$ *is onto, and its kernel is isomorphic to* Λ_w/Λ_r *where* Λ_r *is the root lattice of* $[\mathcal{L},\mathcal{L}]$ *and* Λ_w *is the weight lattice of* M *as an* $[\mathcal{L},\mathcal{L}]$*-module. Thus* $\operatorname{Inn}\operatorname{Aut}\mathcal{L} \subseteq \operatorname{im} r \subseteq \operatorname{im}\overline{r}$.

Thus if $\nu \in \operatorname{Aut}(\overline{S},\sigma)$, $\nu = \eta_1\eta_2$ where $\eta_1 \in \operatorname{Inn}\operatorname{Aut}(S,\sigma)$ and $\eta_2\overline{r}$ stabilizes a Borel subalgebra of $\mathcal{L}$.

Now suppose $\tau \in \operatorname{Aut}\mathcal{L}$ stabilizes a Borel subalgebra B of $\mathcal{L}$. Then it stabilizes the Cartan subalgebra H contained in B, and if $\tau^*\colon H^* \to H^*$ is defined by $h(\mu\tau^*) := \mu(h\tau^{-1})$ for $h \in H$, $\mu \in H^*$, then $\alpha\tau^*$ is a positive root if α is a positive root and $\Delta\tau^* = \Delta$ for Δ a base of the positive roots where $\Delta\tau^* := \{\alpha\tau^* | \alpha \in \Delta\}$.

THEOREM 3.2. *Suppose* $\tau \in \operatorname{Aut}\mathcal{L}$ *stabilizes a Borel subalgebra* B *of* $\mathcal{L}$ *and* N_i *is the* λ_i *weight space relative to* B *for* $\lambda_i \in \Lambda = \{\lambda_1,\dots,\lambda_l\}$. *Then* $\tau \in \operatorname{im}\overline{r}$ *if and only if* $\Lambda\tau^* = \Lambda$ *and* $\dim N_i = \dim N_j$ *if* $\lambda_i\tau^* = \lambda_j$.

PROOF. Suppose $\eta \in \operatorname{Aut}(\overline{S},\sigma)$ with $\eta\overline{r} = \tau$. Then if μ is a weight of M as an $\mathcal{L}$-module and M_μ is the μ weight space of M, $x \in M_{\mu\tau^*}$ for $x \in M_\mu$. Since τ permutes positive root spaces, $x\eta$ is a lowest weight vector if and only if x is a lowest weight vector. Thus $\Lambda\tau^* = \Lambda$ and $\dim N_i = \dim N_j$ if $\lambda_i\tau^* = \lambda_j$.

Conversely, suppose $\tau \in \operatorname{Aut}\mathcal{L}$ stabilizes B, $\Lambda\tau^* = \Lambda$ and $\dim N_i = \dim N_j$ if $\lambda_i\tau^* = \lambda_j$. Define a new action of $\mathcal{L}$ on M by $x \cdot l := x(l\tau)$ and denote M with this new $\mathcal{L}$-action by $\tilde{M}$. Now if $x \in M_\mu$, then $x \in \tilde{M}_{\mu\tau^{-1}}$ so M and $\tilde{M}$ have the same lowest weights. Since the λ_i weight spaces of M and $\tilde{M}$ are the same dimension for $\lambda_i \in \Lambda$, M and $\tilde{M}$ are isomorphic $\mathcal{L}$-modules, i.e. there is a linear bijection $\nu\colon M \to \tilde{M}$ with $(xl)\nu = (x\nu)\cdot l = (x\nu)(l\tau)$. Thus $\eta \in \operatorname{GL}(M \oplus \mathcal{L})$ defined by $(x+l)\eta := x\nu + l\tau$ is in $\operatorname{Aut}(\overline{S},\sigma)$ and $\eta\overline{r} = \tau$. Hence $\tau \in \operatorname{im}(\overline{r})$.

Note 3.3. If $\eta \in \operatorname{Aut}(\overline{S},\sigma)$ with $\eta(\overline{r}) = \tau$ stabilizing a Borel subalgebra B of $\mathcal{L}$, then $y\eta$ is a highest weight vector of M if and only if y is a highest weight vector. In particular, if x_1 and x_2 are nonzero lowest weight vectors with $x_1\eta = x_2$, let $y_1 = x_1 \cdot e_1 \cdot \dots \cdot e_n$ be a nonzero highest weight vector and let y_2 be any nonzero highest weight vector of the irreducible submodule generated by x_2. Then $y_1\eta = (x_1\eta)(e_1\tau)\cdot\dots\cdot(e_n\tau) = cy_2$ for some $c \in k$ and $c \neq 0$ since η is injective.

It is also important to note that if $\{x_{i,1},\dots,x_{i,k}\}$ is a basis of N_i and $\{x'_{j,1},\dots,x'_{j,k}\}$ is a basis of N_j, then the η constructed in the second half of the proof above can be defined so that $x_{i,n}\eta = x'_{j,n}$ for $n = 1,\dots,k$ and $i = 1,\dots,l$. For if $M_{i,n}$ is the irreducible submodule of M generated by $x_{i,n}$ and $\tilde{M}_{j,n}$ is the irreducible submodule of $\tilde{M}$ generated by $x'_{j,n}$, then $M_{i,n}$ and $\tilde{M}_{j,n}$ are isomorphic $\mathcal{L}$-modules since $\lambda_i\tau^* = \lambda_j$. Let $\nu_{i,n}\colon M_{i,n} \to \tilde{M}_{j,n}$ be such an isomorphism. Adjusting

$\nu_{i,n}$ by a scalar multiple if necessary, we may assume $x_{i,n}\nu_{i,n} = x'_{j,n}$. Letting $M_i := \sum_{n=1}^k M_{i,n}$ and $\tilde{M}_j = \sum_{n=1}^k \tilde{M}_{j,n}$, define $\nu_i\colon M_i \to \tilde{M}_j$ by $(\sum_{n=1}^k z_n)\nu_i := \sum_{n=1}^k (z_n\nu_{i,n})$ for $z_n \in M_{i,n}$ and define $\nu\colon M \to \tilde{M}$ by $(\sum_{i=1}^k z_i)\nu := \sum_{i=1}^l (z_i\nu_i)$. As before η is defined by $(x+l)\eta := x\nu + l\tau$.

Note 3.4. Finally note that if $\eta \in \operatorname{Aut}(\overline{\mathcal{S}},\sigma)$, then η is completely determined by $\eta\overline{r} = \tau \in \operatorname{Aut}\mathcal{L}$ and the action of η on a set of vectors $\{x_1,\dots,x_j\}$ spanning the lowest weight spaces of M since vectors of the form $x_i l_1 \cdots l_n$ span M and $(x_i l_1 \cdots l_n) = (x_i\eta)(l\tau)\cdots(l_n\tau)$.

Having determined $\operatorname{im}\overline{r}$, the following lemma will allow us to determine $\operatorname{im} r$ since $\operatorname{im} r \subseteq \operatorname{im}\overline{r}$. Note that the lemma is true for any LMTS.

LEMMA 3.5. *Suppose $\eta \in \operatorname{Aut}(\overline{\mathcal{S}},\sigma)$. Then $\eta \in \operatorname{Aut}(\mathcal{S},\sigma)$ if and only if for all $l \in \mathcal{L}$, $y, z \in M$*

$$b(l\eta, R(y,z)\eta) = \phi((zl)\eta, y\eta). \tag{3.5.1}$$

PROOF. Suppose $\eta \in \operatorname{Aut}(\mathcal{S},\sigma)$. Then $R(y,z)\eta = R(y\eta, z\eta)$ so by (2.2) $b(l\eta, R(y,z)\eta) = b(l\eta, R(y\eta, z\eta)) = \phi((z\eta)(l\eta), y\eta) = \phi((zl)\eta, y\eta)$ giving (3.5.1). Conversely, suppose $\eta \in \operatorname{Aut}(\overline{\mathcal{S}},\sigma)$ satisfies (3.5.1). Then for $u, v \in M$

$$\begin{aligned} b(R(u,v)\eta, R(y,z)\eta) &= \phi([zR(u,v)\eta], y\eta) = \phi((z\eta)[R(u,v)\eta], y\eta) \\ &= b(R(u,v)\eta, R(y\eta, z\eta)) \end{aligned}$$

by (2.2) so $R(y,z)\eta = R(y\eta, z\eta)$ and $\eta \in \operatorname{Aut}(\mathcal{S},\sigma)$.

COROLLARY 3.6. *Suppose $(M, \{\ ,\ ,\ \})$ is basic and $\eta \in \operatorname{Aut}(\overline{\mathcal{S}},\sigma)$. Then $\eta \in \operatorname{Aut}(\mathcal{S},\sigma)$ if and only if*

$$\phi(z\eta, y\eta) = \phi(z, y) \tag{3.6.1}$$

for all $y, z \in M$.

PROOF. If $\mathcal{L}$ is semisimple, then $b(l\tau, l_1\tau) = b(l, l_1)$ for all $\tau \in \operatorname{Aut}\mathcal{L}$, $l, l_1 \in \mathcal{L}$ [**4**]. Thus $b(l\eta, l_1\eta) = b(l, l_1)$ for all $l, l_1 \in \mathcal{L}$ if $(M, \{\ ,\ ,\ \})$ is Type (I) or (II) or if $(M, \{\ ,\ ,\ \})$ is Type (III) with $\mathcal{L}$ semisimple. If $(M\{\ ,\ ,\ \})$ is Type (III) with $\mathcal{L}$ having a one-dimensional center, Theorem 3.2 implies that either $M_1\eta = M_2$ and $M_2\eta = M_1$ or $M_1\eta = M_1$ and $M_2\eta = M_2$. In the first case $c\eta = -c$ and in the second case $c\eta = c$ so $b(c\eta, c\eta) = b(c,c)$ and $b(l\eta, l_1\eta) = b(l, l_1)$ for all $l \in \mathcal{L}$ for $(M, \{\ ,\ ,\ \})$ any basic LMTS. Then (3.5.1) becomes $\phi(zl\eta, y\eta) = b(l\eta, R(y,z)\eta) = b(l, R(y,z)) = \phi(zl, y)$ by (2.2) giving (3.6.1) since M is spanned by vectors of the form zl.

Since $\operatorname{Inn}\operatorname{Aut}\mathcal{L} \subseteq \operatorname{im} r$, we again only need to determine which $\tau \in \operatorname{Aut}\mathcal{L}$ stabilizing a Borel subalgebra B are in $\operatorname{im} r$.

THEOREM 3.7. *Suppose $(M, \{\ ,\ ,\ \})$ is a basic LMTS and $\tau \in \operatorname{im}\overline{r}$ stabilizes a Borel subalgebra B of $\mathcal{L}$.*

(i) *If $\lambda\tau^* = \lambda$ for all $\lambda \in \Lambda$, then $\tau \in \operatorname{im} r$.*

(ii) *If $(M, \{\ ,\ ,\ \})$ is Type* (III) *with $a = \pm 1$ and $\lambda_1\tau^* = \lambda_2$, then $\tau \in \operatorname{im} r$.*

PROOF. We want to find $\eta \in \operatorname{Aut}(\overline{\mathcal{S}},\sigma)$ with $\eta\overline{r} = \tau$ and such that (3.6.1) is satisfied. By Note 3.4 we can do this simply by specifying the images of the lowest weight vectors.

Suppose $\lambda\tau^* = \lambda$. If $(M, \{\ ,\ ,\ \})$ is Type (I) let x be a nonzero lowest weight vector and $y := xe_1 \cdots e_n$ be a nonzero highest weight vector. By Note 3.3, $x(e_1\tau)\cdots(e_\eta\tau) = cy$ for some $c \in k^*$. By Notes 3.3 and 3.4 we can define $\eta \in \mathrm{Aut}(\overline{\mathcal{S}}, \sigma)$ by $l\eta := l\tau$ and $x\eta := \sqrt{c}^{-1}x$. Then $y\eta = \sqrt{c}y$ and $\phi(x\eta, y\eta) = \phi(x, y)$. If w is a weight vector of M of weight $\mu \neq -\lambda$, by (2.1) $\phi(x, w) = 0 = \phi(x\eta, w\eta)$ since $w\eta$ has weight $\mu\tau^* \neq -\lambda$. Thus $\phi(x\eta, w\eta) = \phi(x, w)$ for all $w \in M$. By induction on m

$$\begin{aligned}\phi((xl_1\cdots l_m)\eta, w\eta) &= \phi((x\eta)(l_1\eta)\cdots(l_m\eta), w\eta)\\ &= -\phi((x\eta)(l\eta)\cdots(l_{m-1}\eta), (w\eta)(l_m\eta)) = -\phi(xl_1\cdots l_{m-1})\eta, (wl_m)\eta,\\ &= -\phi(xl_1\cdots l_{m-1}, wl_m) = \phi(xl_1\cdots l_m, w)\end{aligned}$$

so η satisfies (3.6.1) and hence $\eta \in \mathrm{Aut}(\mathcal{S}, \sigma)$ so $\tau \in \mathrm{im}\, r$. If $(M\{\ ,\ ,\ \})$ is Type (II) or (III) Note 3.3 again shows that if x_2 is a nonzero lowest weight vector of M_2 and $y_2 = x_2e_1\cdots e_n$ is a nonzero highest weight vector, then $x_2(e_1\tau)\cdots(e_n\tau) = cy_2$ for some $c \in k^*$. Let x_1 be a nonzero lowest weight vector of M_1 and define $\eta \in \mathrm{Aut}(\overline{\mathcal{S}}, \sigma)$ by $l\eta := l\tau$, $x_1\eta := x_1$ and $x_2\eta := c^{-1}x_2$. Thus $y_2\eta = y_2$ so $\phi(x_1\eta, y_2\eta) = \phi(x_1, y_2)$ and $\phi(y_2\eta, x_1\eta) = \phi(y_2, x_1)$. By the same induction argument as before $\phi(z\eta, w\eta) = \phi(z, w)$ and $\phi(w\eta, z\eta) = \phi(w, z)$ for all $z \in M_1$, $w \in M_2$ giving (3.6.1).

For (ii) let $x_i \in M_i$ be nonzero lowest weight vectors for $i = 1, 2$ and $y_i \in M_i$ be nonzero highest weight vectors for $i = 1, 2$ with $y_2 = x_2e_1\cdots e_n$. Then by Note 3.3, $x_1(e_1\tau)\cdots(e_n\tau) = cy_1$ for some $c \in k^*$. Also $\phi(x_1, y_2) = d\phi(x_2, y_1)$ so if we define $\eta \in \mathrm{Aut}(\overline{\mathcal{S}}, \sigma)$ by $l\eta := l\tau$, $x_1\eta := x_2$, and $x_2\eta := dc^{-1}x_1$, then $y_2\eta = dy_1$ and $\phi(x_1\eta, y_2\eta) = d\phi(x_2, y_1) = \phi(x_1, y_2)$ and again using induction we have $\phi(z\eta, w\eta) = \phi(z, w)$ for all $z \in M_1$, $w \in M_2$. Since ϕ is symmetric or symplectic we also have $\phi(w\eta, z\eta) = \phi(w, z)$ for $z \in M_1$, $w \in M_2$ so η satisfies (3.6.1).

COROLLARY 3.8. *Suppose $(M\{\ ,\ ,\ \}, \mathcal{L}, b, \phi)$ is a basic LMTS and ϕ is symmetric or symplectic. Then* $\mathrm{im}\, r = \mathrm{im}\, \overline{r}$. *Hence* $\mathrm{Aut}(\overline{\mathcal{S}}, \sigma) = \ker \overline{r}\mathrm{Aut}(\mathcal{S}, \sigma)$.

PROOF. The first statement is an immediate consequence of Theorems 3.1, 3.2 and 3.7. For the second one if $\eta \in \mathrm{Aut}(\overline{\mathcal{S}}, \sigma)$, there is an $\eta_1 \in \mathrm{Aut}(\mathcal{S}, \sigma)$ with $\eta\overline{r} = \eta_1 r = \eta_1\overline{r}$ so $\eta\eta_1^{-1} \in \ker \overline{r}$.

Thus the difference between $\mathrm{Aut}(\overline{\mathcal{S}}, \sigma)$ and $\mathrm{Aut}(\mathcal{S}, \sigma)$ lies in $\ker \overline{r}$ if $(M\{\ ,\ ,\ \}, \mathcal{L}, b, \phi)$ is basic and ϕ is symmetric or symplectic. The following theorem finishes the question of determining $\mathrm{im}\, r$ for $(M\{\ ,\ ,\ \})$ a basic LMTS.

THEOREM 3.9. *Suppose $(M\{\ ,\ ,\ \})$ is Type* (III) *with $a \neq \pm 1$ and $\tau \in \mathrm{im}\, \overline{r}$ stabilizes a Borel subalgebra of $\mathcal{L}$. Then $\tau \in \mathrm{im}\, r$ if and only if $\lambda_i\tau^* = \lambda_i$ for $i = 1, 2$.*

PROOF. If $\lambda_i\tau^* = \lambda_i$ for $i = 1, 2$ $\tau \in \mathrm{im}\, r$ by Theorem 3.7(i). If $\lambda_i\tau^* \neq \lambda_i$ for $i = 1, 2$, then $\lambda_1\tau^* = \lambda_2$ and $\lambda_2\tau^* = \lambda_1$ by Theorem 3.2, so if $\eta \in \mathrm{Aut}(\mathcal{S}, \sigma)$ with $\eta r = \tau$, $M_1\eta = M_2$ and $M_2\eta = M_1$. But η must satisfy (3.6.1), so for $y \in M_2$, $z \in M_1$, $\phi(y, z) = \phi(y\eta, z\eta) = a\phi(z\eta, y\eta) = a\phi(z, y) = a^2\phi(y, z)$. Thus $a^2 = 1$, i.e. $a = \pm 1$, a contradiction. Hence no such η exists.

4. Kernels. In this section we will determine $\ker \overline{r}$ and $\ker r$.

LEMMA 4.1. *Suppose* $\Lambda = \{\lambda_1, \dots, \lambda_l\}$, N_i *is the* λ_i *weight space of* M *for* $i = 1, \dots, l$, *and* $P = N_1 \oplus \cdots \oplus N_l$. *Then* N_i *is* $\ker \overline{r}$ *invariant and if* $\dim N_i = n_i$ *for* $i = 1, \dots, l$, *then* $\psi\colon \ker \overline{r} \to \mathrm{GL}(n_1, k) \times \cdots \times \mathrm{GL}(n_l, k)$ *defined by* $\eta\psi := \eta|_p$ *is an isomorphism.*

PROOF. This follows from the proof of Theorem 3.2 and Notes 3.3 and 3.4 since if $\eta \in \ker \overline{r}$, η certainly fixes a Borel subalgebra of $\mathcal{L}$.

LEMMA 4.2. *Suppose* $\eta \in \ker \overline{r}$. *Then* $\eta \in \ker r$ *if and only if* η *satisfies* (3.6.1).

PROOF. If $\eta \in \ker \overline{r}$, $l\eta = l$ for all $l \in \mathcal{L}$, so $b(l\eta, l_1\eta) = b(l, l_1)$ for all l, l_1. The rest of the proof is the same as that of Corollary 3.6 using Lemma 4.1 of [**3**].

LEMMA 4.3. (i) *If* $(M\{\ ,\ ,\ \})$ *is Type* (I), $\ker \overline{r} \cong k^*$ *and* $\ker r \cong C_2$, *the cyclic group of order* 2.

(ii) *If* $(M\{\ ,\ ,\ \})$ *is Type* (II), $\ker \overline{r} \cong \mathrm{GL}(2, k)$ *and* $\ker r \cong \mathrm{SL}(2, k)$.

(iii) *If* $(M\{\ ,\ ,\ \})$ *is Type* (III), $\ker \overline{r} \cong k^* \times k^*$ *and* $\ker r \cong k^*$.

PROOF. The assertions about $\ker \overline{r}$ follow from Lemma 4.1 and those about $\ker r$ for Types (I) and (III) follow immediately from Lemma 4.2. For Type (II) recall that there is an ideal $\mathcal{K}$ of $\mathrm{Der}(M\{\ ,\ ,\ \})$ isomorphic to $\mathrm{sl}(2)$ and consisting of endomorphisms of M which commute with $\mathcal{L}$, so $PD \subseteq P$ for $D \in \mathcal{K}$ and P the lowest weight space of M. Now if $D \in \mathcal{K}$, $\phi(zD, y) = -\phi(z, yD)$ by Lemma 2.3 of [**2**]. Hence if $D \in \mathcal{K}$ is nilpotent, $\exp D \in \mathrm{Aut}(M\{\ ,\ ,\ \})$ where $\exp D := \mathrm{id} + D$, so $\eta_D \in \ker r$ where $(x + l)\eta_D := x(\exp D) + l$. Note that the group generated by $\{\eta_D | D \in \mathcal{K} \text{ nilpotent}\}$ is isomorphic to $\mathrm{SL}(2, k) \subseteq \mathrm{GL}(2, k) = \ker \overline{r}$ under restriction to P. Now $\mathrm{GL}(2, k) = k^*\mathrm{SL}(2)$ where $\eta_b \in k^*$ for $b \in k^*$ is defined by $(x + l)\eta_b := bx + l$. Then Lemma 4.2 gives that $\eta_b \in \ker r$ if and only if $b = \pm 1$.

THEOREM 4.4. *Suppose* $(M\{\ ,\ ,\ \}, \mathcal{L}, b, \phi)$ *is a basic LMTS and* ϕ *is symmetric or symplectic. Then* $\mathrm{Aut}(\mathcal{S}, \sigma)$ *is a normal subgroup of* $\mathrm{Aut}(\overline{\mathcal{S}}, \sigma)$ *and*

$$\mathrm{Aut}(\overline{\mathcal{S}}, \sigma)/\mathrm{Aut}(\mathcal{S}, \sigma) \cong k^*.$$

PROOF. For each type $\ker \overline{r} = k^* \ker r$ where for Types (I) and (III) $\eta_b \in k^*$ is defined for $b \in k^*$ by $(x + l)\eta_b := bx + l$ and for Type (III) η_b is defined by $(x_1 + x_2 + l)\eta_b = bx_1 + x_2 + l$ for $x_i \in M_i$. Hence by Corollary 3.8, $\mathrm{Aut}(\overline{\mathcal{S}}, \sigma) = \ker \overline{r}\mathrm{Aut}(\mathcal{S}, \sigma) = k^*\mathrm{Aut}(\mathcal{S}, \sigma)$. For Types (I) and (II) $\eta_b^{-1}\nu\eta_b = \nu$ for $\nu \in \mathrm{Aut}(\mathcal{S}, \sigma)$ and for Type (III) if $\nu \in \mathrm{Aut}(\mathcal{S}, \sigma)$ with $M_i\nu = M_i$, $i = 1, 2$, then $\eta_b^{-1}\nu\eta_b = \nu$ and if $\nu \in \mathrm{Aut}(\mathcal{S}, \sigma)$ with $M_1\nu = M_2$, $M_2\nu = M_1$, then $\eta_b^{-1}\nu\eta_b = \nu\xi$ where $(x_1 + x_2 + l)\xi := bx_1 + b^{-1}x_2 + l$. Then $\xi \in \mathrm{Aut}(\mathcal{S}, \sigma)$ so $\mathrm{Aut}(\mathcal{S}, \sigma)$ is a normal subgroup in all three cases.

COROLLARY 4.5. *Suppose* $(M\{\ ,\ ,\ \}, \mathcal{L}, b, \phi)$ *is a basic LMTS with* ϕ *symmetric or symplectic. If* $\Lambda_w \subseteq \Lambda_r$, $\mathrm{Aut}(\mathcal{S}, \sigma) \cong \ker \overline{r} \times \mathrm{im}\, \overline{r}$.

PROOF. This follows from Theorem 3.1.

Note 4.5. When studying the forms of a Type (III) LMTS $(M\{\ ,\ ,\ \}, \mathcal{L}, b, \phi)$ in a subsequent paper, we will need to consider the groups $\mathrm{Aut}^*(\overline{\mathcal{S}}, \sigma) := \{\eta \in \mathrm{Aut}(\overline{\mathcal{S}}, \sigma) | M_i\eta = M_i \text{ for } i = 1, 2\}$ and $\mathrm{Aut}^*(\mathcal{S}, \sigma) := \mathrm{Aut}(\mathcal{S}, \sigma) \cap \mathrm{Aut}^*(\overline{\mathcal{S}}, \sigma)$ and also the homomorphisms $\overline{r}^*\colon \mathrm{Aut}^*(\overline{\mathcal{S}}, \sigma) \to \mathrm{Aut}\, \mathcal{L}$ and $r^*\colon \mathrm{Aut}^*(\mathcal{S}, \sigma) \to \mathrm{Aut}\, \mathcal{L}$

given by restriction. The results obtained already give the following statements about these groups: $\ker r^* \cong k^*$, $\ker \bar{r}^* \cong k^* \times k^*$, $\operatorname{Inn}\operatorname{Aut}\mathcal{L} \subseteq \operatorname{im} r^* = \operatorname{im} \bar{r}^*$, and $\operatorname{Aut}^*(\mathcal{S},\sigma)$ is a normal subgroup of $\operatorname{Aut}^*(\mathcal{S},\sigma)$ with $\operatorname{Aut}^*(\overline{\mathcal{S}},\sigma)/\operatorname{Aut}^*(\mathcal{S},\sigma) \cong k^*$. If $\eta \in \operatorname{Aut}^*(\overline{\mathcal{S}},\sigma)$ such that η stabilizes a Borel subalgebra B, then $\lambda_i\eta^* = \lambda_i$ for $i = 1,2$. Either $\operatorname{Aut}^*(\overline{\mathcal{S}},\sigma) = \operatorname{Aut}(\overline{\mathcal{S}},\sigma)$ or $\operatorname{Aut}^*(\overline{\mathcal{S}},\sigma)$ is a normal subgroup of $\operatorname{Aut}(\overline{\mathcal{S}},\sigma)$ of index 2. If $a \neq \pm 1$, $\operatorname{Aut}^*(\mathcal{S},\sigma) = \operatorname{Aut}(\mathcal{S},\sigma)$.

References

1. N. C. Hopkins, *Decomposition of Lie module triple systems*, Algebras, Groups and Geometries (to appear).
2. ——, *On the derivation algebras of Lie module triple systems*, J. Algebra (to appear).
3. ——, *Some structure theory for a class of triple systems*, Trans. Amer. Math. Soc. **289** (1985), 203–212.
4. N. Jacobson, *Lie algebras*, Dover, New York, 1962.
5. W. Lister, *A structure theory of Lie triple systems*, Trans. Amer. Math. Soc. **72** (1952), 217–242.
6. R. Steinberg, *Lectures on Chevalley group*, Yale Univ. Notes, New Haven, Conn., 1967.

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