ABSTRACT. For $1 \leq p < \infty$, $\Omega$ an open and bounded subset of $\mathbb{R}^n$, and a nonincreasing and nonnegative function $\varphi$ defined in $(0, \rho_0)$, $\rho_0 = \text{diam} \Omega$, we introduce the space $M^p_{\varphi,0}(\Omega)$ of locally integrable functions satisfying
\[
\inf_{c \in \mathbb{C}} \left\{ \int_{B(x_0,\rho) \cap \Omega} |f(x) - c|^p \, dx \right\} \leq A \rho \varphi^p(\rho)
\]
for every $x_0 \in \Omega$ and $0 < \rho \leq \rho_0$, where $|B(x_0, \rho)|$ denotes the volume of the ball centered in $x_0$ and radius $\rho$. The constant $A > 0$ does not depend on $B(x_0, \rho)$.

(i) We list some results on the structure, regularity, and density properties of the space so defined.

(ii) $M^p_{\varphi,0}$ is represented as the dual of an atomic space.

Given an open and bounded subset $\Omega$ of $\mathbb{R}^n$, let $\rho_0 = \text{diam} \Omega$, $0 \leq \lambda \leq n$, $1 \leq p < \infty$. We denote by $L^{p,\lambda}(\Omega)$ the space introduced by Morrey of locally integrable functions $f(x)$ for which there is a constant $A = A(f) > 0$, such that
\[
\int_{B(x_0,\rho) \cap \Omega} |f(x)|^p \, dx \leq A \rho^\lambda
\]
for every $x_0 \in \Omega$, and $0 < \rho \leq \rho_0$, where $B(x_0, \rho) = \{x \in \mathbb{R}^n / |x - x_0| < \rho\}$.

More generally, given $k \geq 0$, we say that $f(x)$ belongs to $L^{p,k}(\Omega)$ if there is a constant $A > 0$ such that for every $x_0 \in \Omega$, and $0 < \rho \leq \rho_0$ we have
\[
\inf_{P \in \mathbb{P}_k} \left\{ \int_{B(x_0,\rho) \cap \Omega} |f(x) - P(x)|^p \, dx \right\} \leq A \rho^\lambda,
\]
where $\mathbb{P}_k$ is the class of polynomials of degree $\leq k$.

We observe that $L^{p,0}(\Omega)$ describes $L^p(\Omega)$ and that $L^{p,n}(\Omega)$ is a slightly more restricted version of BMO, the space of functions of bounded mean oscillation.

More generally, we can replace the second member in (1) by $|B(x_0, \rho)| \varphi^p(\rho)$, where $|\cdot|$ denotes the Lebesgue measure and $\varphi$ is a function from $(0, \rho_0]$ into $[0, \infty)$. Let $M^{p}_p(\Omega)$ be the space so defined. In the same way, using (2) we define the space $M^{p}_{\varphi,k}(\Omega)$.

Condition (1) is obtained with $\varphi(t) = t^{(\lambda - n)/p}$. So the Morrey space is related to some nonincreasing function $\varphi(t)$ such that $\varphi(t) \to \infty$ when $t \to 0$. This is the kind of functions we will consider here.

Let us set
\[
\|f\| = \|f\|_{L^p(\Omega)} + \inf_{A \in C} A^{1/p}.
\]
This is a norm in $M^p_{\varphi, k}(\Omega)$ and in $M^p_{\varphi}(\Omega)$, so we get a Banach space.

Though $R^n$ is not bounded, in the same manner we can define $M^p_{\varphi}(R^n)$ and $M^p_{\varphi, k}(R^n)$. In this case we use $M^p_{\varphi}$ and $M^p_{\varphi, k}$.

We list some results on the structure, regularity, and density properties of the generalized Morrey space.

**Proposition 1.** Let $\Omega$ be an open and bounded subset of $R^n$. Suppose that there exists a constant $B > 0$ such that

$$|B(x_0,p)| > B t^n$$

for all $x_0 \in \Omega$, $\rho \leq \rho_0$.

Let $\varphi(t)$ satisfy the following conditions:

(i) There is $0 < D < 1$ such that $\varphi(2t) \leq D \varphi(t)$ for $0 < t \leq \rho_0$.

(ii) $\varphi(t)$ is nonincreasing and $t^n \varphi(t)$ is nondecreasing.

Then, $M^p_{\varphi}(\Omega)$ and $M^p_{\varphi, k}(\Omega)$ describe the same space.

**Proof.** This result has been proved by Campanato [2] when $\varphi(t) = t^{(\lambda-n)/p}$, $0 < \lambda < n$. The same proof applies in the general situation with minor changes.

Condition $0 < D < 1$ imposed in (i) implies that $\varphi(t)$ cannot be a constant function. In fact, in that case (1) would mean that $f(x)$ belongs to $L^\infty(\Omega)$ which is not true for every function in BMO.

Functions in $M^p_{\varphi}(\Omega)$ can be trivially extended to $R^n$.

**Proposition 2.** We suppose that $t^n \varphi(t)$ is nondecreasing in $(0, \infty)$. Then, given $f(x) \in M^p_{\varphi}(\Omega)$, its extension $f^*(x)$ defined as zero outside $\Omega$ belongs to $M^p_{\varphi}$.

**Proof.** It is simple. For $x_0 \in R^n$ and $\rho > 0$ it suffices to consider several cases: whether $x_0$ belongs to $\Omega$ or not and $\rho \leq \rho_0$ or not.

This result is also true for the space $M^p_{\varphi, k}$ and includes the case $\varphi(t) = t^{(\lambda-n)/p}$ for $\lambda \geq 0$.

We can deduce that the definition of BMO as $L^p_0(\Omega)$ is more restricted than the definition given in [3]. In fact, function $\log t$ belongs to BMO($R^+$) but its extension as zero for $t \leq 0$ does not belong to BMO.

From the estimate obtained by John and Nirenberg in [3] for the distribution function in BMO, we deduce that when $\lambda = n$ and $k = 0$, condition (2) is satisfied independently of the exponent $p$ which we have used. This is no longer true for $0 < \lambda < n$, because as was proved in [1], we can get functions in $M^p_{\varphi}(\Omega)$ with a distribution function arbitrarily large.

Functions in $M^p_{\varphi}(\Omega)$ cannot be approximated by functions in $C^\infty(\Omega)$, nor even by continuous functions. In fact, we find a simple example in $L^{p,\lambda}(\Omega)$, $0 < \lambda < n$ : for $x_0 \in \Omega$ and $\rho_1 \leq \rho_0$ such that $B(x_0, \rho_1) \subset \Omega$, if we define

$$f_{x_0}(x) = |x-x_0|^{(\lambda-n)/p}, \quad x \in \Omega,$$

then we obtain

$$\|f - h\| \geq 2^{-p-1}|S^{n-1}|$$

for any continuous function $h(x)$ in $\Omega$, where $S^{n-1} = \{x \in R^n/|x| = 1\}$. To see this it suffices to find $0 < \rho \leq \rho_1$ such that

$$\int_{B(x_0, \rho)} |f_{x_0}(x) - h(x)|^p \, dx \geq 2^{-p-1}|S^{n-1}|\rho^\lambda.$$
If $M = \sup_{x \in B(x_0, \rho_1)} |h(x)|^p$, then

$$
\int_{B(x_0, \rho)} |f_{x_0}(x) - h(x)|^p \, dx \geq 2^{-p} \int_{B(x_0, \rho)} |f_{x_0}(x)|^p \, dx - \int_{B(x_0, \rho)} |h(x)|^p \, dx \\
\geq 2^{-p} |S^{n-1}|^{\rho^\lambda - M} |S^{n-1}|^{\rho^n} = |S^{n-1}|^{\rho^\lambda (2^{-p} - M \rho^{n-\lambda})}.
$$

Then it suffices to take $0 < \rho \leq \rho_1$ such that $(2^{-p} - M \rho^{n-\lambda}) \geq 2^{-p-1}$.

The situation changes when we consider the following subset of $M^P_\varphi$:

$$
\overline{M}^P_\varphi = \{ f \in M^P_\varphi \text{ such that } \| f(x - y) - f(x) \| \to 0 \text{ when } y \to 0 \}.
$$

We have

**Proposition 3.** Let $\varphi(t)$ be nonincreasing such that $t^n \varphi(t)$ is nondecreasing. Let $\psi(x) \in C^\infty(R^n)$ be supported in $B(0,1)$, $\int \psi(x) \, dx = 1$, $0 \leq \psi(x) \leq 1$, and $\psi(x) = j^n \psi(jx)$. Then

(i) If $f(x) \in \overline{M}^P_\varphi$, $f * \psi_j(x) \to f(x)$ in the $M^P_\varphi$ norm as $j \to \infty$.

(ii) If $f(x)$ can be approximated by functions in $C^1_0$, then $f(x) \in \overline{M}^P_\varphi$.

**Proof.** (i) for $\rho > 0$, $\varepsilon > 0$

$$
\left[ \int_{B(x_0, \rho)} |f * \psi_j(x) - f(x)|^p \, dx \right]^{1/p} \\
\leq \left[ \int_{|y| \leq 1/j} \psi_j(y) \, dy \int_{B(x_0, \rho)} |f(x - y) - f(x)|^p \, dx \right]^{1/p} \\
\leq \varepsilon \| f \|_{\rho^n/p \varphi(\rho)} \text{ if } j \text{ is large.}
$$

(ii) Let $\rho > 0$, $x_0 \in R^n$, $\varepsilon > 0$. Let $g(x) \in C^1_0(R^n)$ such that $\| f - g \| \leq \varepsilon/3$. Then

$$
(*) \quad \left[ \int_{B(x_0, \rho)} |f(x - y) - f(x)|^p \, dx \right]^{1/p} \leq \left[ \int_{B(x_0, \rho)} |f(x - y) - g(x - y)|^p \, dx \right]^{1/p} \\
+ \left[ \int_{B(x_0, \rho)} |g(x) - f(x)|^p \, dx \right]^{1/p} \\
+ \left[ \int_{B(x_0, \rho)} |g(x - y) - g(x)|^p \, dx \right]^{1/p} \\
\leq c2\varepsilon/3\rho^n/p \varphi(\rho) + \int_{B(x_0, \rho)} |g(x - y) - g(x)|^p \, dx \right]^{1/p}.
$$

Let $d > 1$ such that $\text{supp}(g) \subset B(0, d - 1)$. If we take $|y| < 1$, then

$$
\leq c2\varepsilon/3\rho^n/p \varphi(\rho) + |y| \| \nabla g \|_{\infty} B(0, \rho) \right]^{1/p} \\
\leq c2\varepsilon/3\rho^n/p \varphi(\rho) + |y| \| \nabla g \|_{\infty} c' d^n/p \varphi(d).
$$
For $\rho \geq d$, since $t^n \varphi^p(t)$ is nondecreasing, it suffices to take

$$|y| \leq \frac{\varphi(d)\varepsilon/3}{\|\nabla g\|_\infty \rho'}. \tag{*}$$

If $\rho < d$, since $\varphi(t)$ is nonincreasing, we get

$$\leq c2\varepsilon/3\rho^{n/p}\varphi(\rho) + |y|\|\nabla g\|_\infty \rho^{n/p} \tag{**}$$

as in the other case.

Proposition 3 is also true if we replace $\mathbb{R}^n$ for any $\Omega \subset \mathbb{R}^n$, open and bounded.

**Description of $M_{\varphi,0}$ as a dual space.** Fefferman and Stein have characterized BMO as the dual of the Hardy space $H^1$. $M_{\varphi,0}^p$ can also be viewed as the dual of an “atomic” space.

For $1 < p < \infty$, and $B(x_0, \rho)$, we define $A_{B(x_0, \rho)}^p$ as the set of functions $a(x)$ such that

(i) $\text{supp}(a) \subset B(x_0, \rho)$,

(ii) $\int a(x) \, dx = 0$,

(iii) $\|a\|_p \leq 1/|B(x_0, \rho)|^{1/q}\varphi(\rho), \ 1/p + 1/q = 1$.

Let $A_{B(x_0, \rho)}^p = \bigcup_{B(x_0, \rho)} A_{B(x_0, \rho)}^p$. We define $H_{\varphi}^p$ as the set of functions $f(x)$ such that $f(x) = \sum_{i \geq 0} \lambda_i a_i(x)$ in the sense of distributions, where $\lambda_i \in R$, $a_i(x) \in A_{\varphi}^p$, and $\sum_{i \geq 0} |\lambda_i| < \infty$.

We define a norm in $H_{\varphi}^p$

$$\|f\|_{H_{\varphi}^p} = \inf \left( \sum_{i \geq 0} |\lambda_i| \right).$$

Infimum if taken over all atomic decompositions of $f$.

In this section we only consider real valued functions.

**Proposition 4.** Let $1 < p < \infty$, $1/p + 1/q = 1$. If $\varphi(t)$ is nonincreasing and $t^n \varphi^p(t)$ is nondecreasing, then $H_{\varphi}^p$ is a Banach space.

**Proof.** Clearly $\| \|_{H_{\varphi}^p}$ defines a norm in $H_{\varphi}^p$. We must only see that $H_{\varphi}^p$ is complete.

If $\{f_n\}$ is a Cauchy sequence, we can choose a subsequence $\{f_{n_k}\}$ such that $\|f_{n_k} - f_{n_{k-1}}\|_{H_{\varphi}^p} \leq 2^{-k}$. We define

$$f = f_{n_1} + \sum_{k \geq 2} (f_{n_k} - f_{n_{k-1}}).$$

Let $f_{n_k} - f_{n_{k-1}} = \sum_{i \geq 0} \lambda_i^k a_i^k$ be the atomic decomposition such that

$$\sum_{i \geq 0} |\lambda_i^k| \leq \|f_{n_k} - f_{n_{k-1}}\|_{H_{\varphi}^p} 2^{-k}.$$

Then $\sum_{k \geq 0} \sum_{i \geq 0} |\lambda_i^k| < \infty$ and we have a decomposition for $f$, except the series converges in the $H_{\varphi}^p$ norm. But this convergence also takes place in the sense of distributions.
If $\psi$ is a testing function supported in $B(x_1, \rho_1)$ and $a$ is an atom, we have

$$\left| \int a(x)\psi(x) \, dx \right| \leq \|a\|_p \|\psi\|_q \leq \frac{1}{|B(x_1, \rho)|^{1/q} \varphi(\rho)} \left( \int_{B(x_1, \rho) \cap B(x_1, \rho_1)} |\psi(x)|^q \, dx \right)^{1/q}.$$  

If $\rho \leq \rho_1$, then $\varphi(\rho) \geq \varphi(\rho_1)$ and

$$\left| \int a(x)\psi(x) \, dx \right| \leq \frac{1}{|B(x_1, \rho)|^{1/q} \varphi(\rho)} \|\psi\|_\infty |B(x_1, \rho)|^{1/q} \varphi(\rho_1).$$

If $\rho \geq \rho_1$, then $\rho_1^{n/q} \varphi(\rho_1) \leq \rho^{n/q} \varphi(\rho)$ and

$$\left| \int a(x)\psi(x) \, dx \right| \leq \frac{1}{|B(x_1, \rho_1)|^{1/q} \varphi(\rho_1)} \|\psi\|_\infty |B(x_1, \rho_1)|^{1/q} \varphi(\rho_1).$$

Then, the result follows for any $g(x) \in H^{p, \varphi}$.

Consequently, since $f_n$ converges to $f \in H^{p, \varphi}$ Proposition 4 is proved.

Finally, we have the following duality result.

**Proposition 5.** Let $\varphi(t) \geq 0$, $1 < p < \infty$, and $1/p + 1/q = 1$. For any $L \in (H^{p, \varphi})^*$ there exists $g \in M^q_{\varphi, 0}$ such that if $h(x) \in H^{p, \varphi}$ we have

$$L(h) = \int g(x)h(x) \, dx.$$  

Moreover, if $f \in M^q_{\varphi, 0}$ and $h \in H^{p, \varphi}$, then $\int f(x)h(x) \, dx$ is an element of $(H^{p, \varphi})^*$.

**Proof.** The last statement is simple. For $f \in M^q_{\varphi, 0}$ and $a$ an atom of $A^p_{B(x_0, \rho)}$ we have

$$\int a(x)f(x) \, dx = \int a(x)(f(x) - c(x_0, \rho)) \, dx,$$

where $c(x_0, \rho)$ is the constant for which the infimum in (2) is attained. Then

$$\int a(x)f(x) \, dx \leq \|a\|_p \left( \int_{B(x_0, \rho)} |f(x) - c(x_0, \rho)|^q \, dx \right)^{1/q} \leq \frac{1}{|B(x_0, \rho)|^{1/q} \varphi(\rho)} \|f\| \|B(x_0, \rho)|^{1/q} \varphi(\rho).$$

For any $h \in H^{p, \varphi}$ the affirmation follows immediately.

To prove the other statement we see first of all that $(H^{p, \varphi})^* \subset L^q_{\text{loc}}$.

Let $L \in (H^{p, \varphi})^*$. Let $B_k$ be an increasing sequence of balls which cover $R^n$. Let $T_k$ be the restriction operator from $R^n$ to $B_k$. Then $L \circ T_k$ belongs to $(L^p_{0}(B_k))^*$, where $L^p_{0}(B_k)$ denotes the subspace of $L^p(B_k)$ of functions having mean value zero. In fact, if $f \in L^p_{0}(B_k)$, then

$$|L(f)| \leq \|L\|_{(H^{p, \varphi})^*} \|f\|_{H^{p, \varphi}} \leq \|L\|_{(H^{p, \varphi})^*} \|f\|_p \|B(x_0, \rho)|^{1/q} \varphi(\rho).$$

Since $(L^q_{0}(B_k))^* = L^q(B_k)/C(B_k)$ ($C(B_k)$ is the space of the functions that are constant on $B_k$) there exists $g_k \in L^q(B_k)$ such that

$$L(f) = \int f(x)g_k(x) \, dx.$$
Since the $B_k$ are increasing we have $T_k(g_{k+1}) = g_k$. This implies the existence of a function $g \in L^q_{\text{loc}}$.

Now, we must prove that if $g \in L^q_{\text{loc}}$ is in $(H^{\rho,\varphi})^*$, then $g$ belongs to $M_{\varphi,0}^q$. To see this, we use a constant $c(x_0, \rho)$ for which
\[
|\{x \in B(x_0, \rho)/g(x) < c(x_0, \rho)\}| \leq 1/2|B(x_0, \rho)|,
\]
\[
|\{x \in B(x_0, \rho)/g(x) > c(x_0, \rho)\}| \geq 1/2|B(x_0, \rho)|,
\]
and suppose without loss of generality that
\[
\int_{B(x_0, \rho) \cap \{g(x) > c(x_0, \rho)\}} |g(x) - c(x_0, \rho)|^q \, dx
\]
\[
\geq \int_{B(x_0, \rho) \cap \{g(x) \leq c(x_0, \rho)\}} |g(x) - c(x_0, \rho)|^q \, dx.
\]
To simplify, we denote $A = B(x_0, \rho) \cap \{g(x) > c(x_0, \rho)\}$ and $B = B(x_0, \rho) \cap \{g(x) \leq c(x_0, \rho)\}$.

We define an atom $a(x)$ supported in $B(x_0, \rho)$ so
\[
a(x) = \begin{cases} 
[g(x) - c(x_0, \rho)]^{q-1} & \text{for } x \in A, \\
C & \text{in } B(x_0, \rho) \setminus A,
\end{cases}
\]
where $C$ is a constant chosen so that the mean value of $a(x)$ over $B(x_0, \rho)$ is zero.

We have
\[
\int_{B(x_0, \rho)} |g(x) - c(x_0, \rho)|^q \, dx \leq 2 \int_A |g(x) - c(x_0, \rho)|^q \, dx
\]
\[
= 2 \int_A (g(x) - c(x_0, \rho))a(x) \, dx
\]
\[
\leq 2 \int g(x)a(x) \, dx \leq 2 \|g\|_{(H^{\rho,\varphi})^*} \|a\|_{H^{\rho,\varphi}}.
\]

Now,
\[
\|a\|_{H^{\rho,\varphi}} \leq \|a\|_p |B(x_0, \rho)|^{1/q}\varphi(\rho)
\]
\[
\leq |B(x_0, \rho)|\varphi(\rho) \left[ \frac{1}{|B(x_0, \rho)|} \int |a(x)|^p \, dx \right]^{1/p}
\]
\[
\leq |B(x_0, \rho)|\varphi(\rho) \left[ \frac{1}{|B(x_0, \rho)|} \int_A |g(x) - c(x_0, \rho)|^{p(q-1)} \, dx + \frac{1}{|B(x_0, \rho)|} \int_B C^p \, dx \right]^{1/p}.
\]
But
\[
\frac{1}{|B(x_0, \rho)|} \int_B C^p \, dx \leq \left[ |B|^{-1} \int_B C \, dx \right]^p
\]
\[
= \left[ |B|^{-1} \int_A |g(x) - c(x_0, \rho)|^{q-1} \, dx \right]^p
\]
\[
\leq |B|^{-1} \int_A |g(x) - c(x_0, \rho)|^q \, dx.
\]
Hence
\[
\|a\|_{H^{\rho,\varphi}} \leq |B(x_0, \rho)|\varphi(\rho) \left[ \frac{1}{|B(x_0, \rho)|} \int_A |g(x) - c(x_0, \rho)|^q \, dx \right]^{1/p}.
\]
Then
\[
\left[ \int_{B(x_0, \rho)} |g(x) - c(x_0, \rho)|^q \, dx \right]^{1/q} \leq c |B(x_0, \rho)|^{1/q} \varphi(\rho).
\]
So, Proposition 5 is proved.

REFERENCES


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