APPROXIMATION BY RATIONAL FUNCTIONS

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ABSTRACT. Making use of the Hardy-Littlewood maximal function, we give a new proof of the following theorem of Pekarski: If $f'$ is in $L \log L$ on a finite interval, then $f$ can be approximated in the uniform norm by rational functions of degree $n$ to an error $O(1/n)$ on that interval.

It is well known that approximation by rational functions of degree $n$ can produce a dramatically smaller error than that for polynomials of degree $n$. The best example of this is Newman's theorem [3] which shows that the function $f(x) = |x|$ can be approximated on $[-1,1]$ by rational functions of degree $n$ to an error $O(\exp(-c\sqrt{n}))$, whereas for polynomials of degree $n$ the error is known to be larger than $c/n$. Other authors have shown that such improvement also occurs for certain classes of functions. For example, V. Popov [5] showed that if $f' \in L^p[0,1]$, with $p > 1$, then $r_n(f) = O(n^{-1})$ where $r_n(f)$ is the error in approximating $f$ by rational functions $R$ of degree at most $n$ in the uniform norm:

$$r_n(f) := \inf_{\deg(R) = n} \|f - R\|_{[0,1]}.$$

To obtain this order of approximation for polynomials requires roughly speaking that $f' \in L_{\infty}$. A striking limiting version of Popov's result was given by A. A. Pekarski [4], who showed that the same conclusion holds when $f' \in L \log L$, i.e. if $|f'| \log(1 + |f'|)$ is integrable.

The Popov and Pekarski proofs of these theorems are quite technical, and it was the purpose of [2] to introduce an elementary technique using maximal functions and partitions of unity for rational functions in order to give a simpler proof of Popov's results. The point of this note is to show that a modification of the technique in [2], albeit a little tricky, will also prove Pekarski's theorem.

The idea in [2] is to partition $[0,1]$ into a set $I$ of disjoint intervals $I$ and construct associated rational functions $\psi_I$ which form a partition of unity: $\sum_{I \in I} \psi_I \equiv 1$. Our rational approximation $R$ is then given by

$$R(x) := \sum_{I \in I} f(x_I) \psi_I(x)$$

with $x_I$ the center of $I$. Of course, the intervals $I$ depend on $f$.

The rational functions $\psi_I$ are constructed using a standard method for partitions of unity. Namely, $\psi_I := \phi_I / \Phi$ with $\Phi := \sum \phi_I$. In the case of Popov's theorem, the $\phi_I$ depend only on the interval $I$ and all can be taken of degree 4. The intervals

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I are determined by using the Hardy-Littlewood maximal functions $M$ which is defined for $g \in L_1$ by

$$Mg(x) := \sup_{J \ni x} \frac{1}{|J|} \int_J |g|,$$

where the sup is taken over all intervals $J \subset [0,1]$ which contain $x$.

To prove the Pekarski theorem, we will need to let the degree of $\phi_I$ depend on $f$. The desired properties of $c_p$ are given in the following lemma.

**Lemma 1.** For each even integer $m \geq 8$, and each interval $I$ there is a nonnegative rational function $c_p$ of degree at most $6m$ with the following properties:

(i) $c_p(x) > 1$, $x \in I$,

(ii) $c_p(x) < 8 \cdot 2^{-\sqrt{n}/4}$, if $2^{-m/\lambda}|I| \leq \text{dist}(x,I) \leq 1/2$ and $0 < \lambda < m$,

(iii) $c_p(x) < 4(a^2 + 1)^{-m}$, if $\text{dist}(x,I) \geq a|I|$ and $a > 0$.

We postpone the proof of this lemma until the end of the paper. We now use this result to prove the following.

**Theorem.** There is an absolute constant $c > 0$ such that for $n = 1,2,\ldots$

$$r_n(f) \leq c \|M(f')\|_{L^1}^{-1}, \quad n = 1, 2, \ldots,$$

whenever $M(f')$ is in $L_1[0,1]$.

**Remark:** It is well known (see e.g. [1]) that $g \in L \log L$ is equivalent to $M(g) \in L_1$ and therefore this theorem is equivalent to Pekarski’s.

**Proof.** It is enough to consider functions $f$ with $\|M(f')\|_1 = 1$. It follows that $\|f'\|_1 \leq 1$ and hence there is a collection $I$ of at most $n$ intervals $I$ which are a disjoint partition of $[0,1]$ and satisfy

(2) $\frac{1}{n} \leq \int_I |f'| \leq \frac{2}{n}, \quad I \in I$.

For each $I \in I$, we let $m_I$ be the smallest integer which is both larger than 7 and also larger than $4n \int_I M(f')$. If $\phi_I$ is the function of Lemma 1 for the interval $I$ and for $m = m_I$, we let $\Phi := \sum_{I \in I} \phi_I$. By Lemma 1, $\Phi \geq 1$, on $[0,1]$ and hence the functions $\psi_I$ satisfy

(3) $\psi_I(x) \leq \phi_I(x), \quad 0 \leq x \leq 1$.

We now take $R$ as in (1) with $x_I$ the center of $I$. Since $\sum m_I \leq 16n$, $R$ has degree $\leq 96n$. To estimate $|f(x) - R(x)|$, we let $I_0$ denote the interval of $I$ which contains $x$; $I_1$ the interval of $I$ immediately to the right of $I_0$; $I_{-1}$ the interval immediately to the left of $I_0$; and so on. We have

(4) $f(x) - R(x) = \sum_{I \in I} (f(x) - f(x_I))\psi_I(x) =: \sum_{-1} + \sum_0 + \sum_1$

Where $\sum_{-1}$ denotes the sum over those $I = I_k$ with $k < -1$, $\sum_1$ the sum over those $I = I_k$ with $k > 1$ and $\sum_0$ the sum of the terms $k = -1,0,1$. Clearly, $|f(x) - f(x_{I_k})| \leq 2(|k| + 1)/n$. Since the $\psi_I$ are nonnegative and add up to one, we have

(5) $\sum_0 \leq 12/n$. 

The estimates for \( \sum_{-1} \) and \( \sum_{1} \) are the same and therefore we estimate only \( \sum_{1} \). For this, we fix \( k > 1 \) and estimate the term in \( \sum_{1} \) corresponding to \( I = I_k \).

We have

\[
e_k := |f(x) - f(x_I)|\psi_I(x) \leq \frac{2(k + 1)}{n} \psi_I(x) \leq \frac{4k}{n} \phi_I(x).
\]

We write \( \text{dist}(x, I) =: a|I| \), with \( a \geq 0 \), and we consider three cases.

Case \( a \geq \sqrt{k} \). Then since \( m \geq 8 \), by (iii) of Lemma 1, we have \( \psi_I(x) \leq \phi_I(x) \leq 4k^{-4} \) and consequently

\[
e_k \leq 16k^{-3}n^{-1}.
\]

Case \( 1/2 \leq a < \sqrt{k} \). The smallest interval \( J \) which contains \( x \) and \( I \) has length \( (a + 1)|I| \) and on \( I \),

\[
M(f') \geq \frac{1}{|J|} \int_J |f'| \geq \frac{k}{n(a + 1)|I|}
\]

and therefore \( m \geq 4n \int_J M(f') \geq 4k/(a + 1) \geq \sqrt{k} \). This gives by (iii) of Lemma 1,

\[
e_k \leq \frac{4k}{n} \phi_I(x) \leq \frac{4k}{n} (a^2 + 1)^{-m} \leq \frac{16k}{n} (5/4)^{-\sqrt{k}}.
\]

Case \( 0 < a < 1/2 \). We write \( a =: 2^{-m/\lambda} \) with \( 0 < \lambda \leq m \). Similar to the second case, for \( u \in I \), we have \( M(f')(u) \geq (k - 1)/n(u - x) \). Therefore,

\[
m \geq 4n \int_I M(f') \geq 4(k - 1) \int_{2^{-m/\lambda}|I|}^{1/2} \frac{du}{u} \geq 2k \left( \frac{m}{\lambda} \right) \log 2.
\]

This shows that \( \lambda \geq 2k \log 2 \geq k \). Hence by (ii) of Lemma 1, we have

\[
e_k \leq \frac{4k}{n} \phi_I(x) \leq \frac{32k}{n} 2^{-\sqrt{k}/4}.
\]

The estimates (7)-(9) serve to show that \( \sum_1 = \sum e_k \leq c n^{-1} \), with \( c \) an absolute constant. This combined with (5) and the corresponding estimate for \( \sum_{-1} \) when placed in (4) proves the theorem.

We turn now to the proof of Lemma 1. For this, we shall use the following:

**LEMMA 2.** For each even integer \( m \geq 8 \) there is a rational function \( R \) of degree \( \leq 2m \) with the following properties:

(i) \( R(x) \geq 1, \ x \in [-1, 0] \),

(ii) \( 0 \leq R(x) \leq 2 \), for \( -\infty < x < \infty \),

(iii) \( |R(x)| \leq 2 \cdot 2^{-m/4j}, \) if \( 2^{-(j+1)} \leq x \leq 1/2, \) with \( \sqrt{m} - 1 \leq j < m \).

**PROOF.** With \( a := 2^{-1/m} \) and \( a_k := a^{k^2} \), we define \( p(x) := \prod_{1}^{m}(x + a_k) \). We first estimate \( \pi(x) := p(-x)/p(x) = \prod_{1}^{m}(-x + a_k)/(x + a_k) \) when \( x \geq 0 \). Since each term in \( \pi \) has absolute value at most 1, we have

\[
|\pi(x)| \leq 1, \quad x \geq 0.
\]

When \( a_m \leq x \leq 1/2 \), we take \( j \) so that \( a_j+1 \leq x \leq a_j \); so \( \sqrt{m} - 1 \leq j < m \). Then,

\[
|\pi(x)| \leq \pi_1(x) := \prod_{1}^{j} \frac{a_k - x}{a_k + x}.
\]
We now use the inequality \((1 - t)/(1 + t) \leq e^{-2t}\), which is valid for \(0 \leq t \leq 1\). This gives

\[
|\pi(x)| \leq \prod_{1}^{j} \frac{1 - x/a_k}{1 + x/a_k} \leq \exp \left( -2 \sum_{1}^{j} \frac{a_{j+1}}{a_k} \right) =: \exp(-2\sigma(j)).
\]

Since \((j + 1)^2 - k^2 \leq (j - k + 1)(2j + 1)\), we have with \(b := a^{2j+1}\),

\[
\sigma(j) \geq \sum_{1}^{j} b^{2^r} = b^{1 - b^j}. \tag{11}
\]

But, since \(\sqrt{m} - 1 \leq j < m\), \(b \geq 1/4\); \(1 - b^j \geq 1/2\); also \(1 - e^{-t} \leq t\), for \(0 < t \leq 1\). Hence,

\[
\sigma(j) \geq \frac{m}{8(2j + 1)\log 2}. \tag{12}
\]

Since \(2\log 2 \leq 1/\log 2\), using our last estimate for \(\sigma(j)\) in (11) gives

\[
|\pi(x)| \leq \exp \left( -\frac{m \log 2}{2(2j + 1)} \right) \leq 2^{-m/8j}, \quad a_{j+1} \leq x \leq a_j, \text{ for } \sqrt{m} - 1 \leq j < m. \tag{12}
\]

We can now take \(R(x) := 2\pi^2(x)/(1 + \pi^2(x))\). Since \(\pi(-x) = 1/\pi(x)\) and \(R(-x) = 2/(1 + \pi^2(x))\), (i) follows from (10). The estimate (ii) is obvious, while (iii) follows immediately from (12).

**Proof of Lemma 1.** It is enough to consider \(I = [-1,0]\) since the lemma then follows for any other interval by a change of scale. We let

\[
T(x) := ((x + 1/2)^2 + 3/4)^{-m}
\]

and \(R\) be as in Lemma 2. We can then take \(\phi(x) := R(x)R(-1 - x)T(x)\). Since \(T(x) \geq 1, x \in I\), (i) follows from (i) of Lemma 2. Since \(T(x) \leq 1, x \notin I\), (ii) follows when \(\lambda \leq 4\) from Lemma 2(ii). For the other values of \(\lambda\), we choose \(j\) so that \(j^2 < m^2/\lambda < (j + 1)^2\), and then (ii) follows from Lemma 2(ii), (iii). Finally, if \(\text{dist}(x, I) \geq a\), then \((x - 1/2)^2 + 3/4 \geq (a + 1/2)^2 + 3/4 \geq a^2 + 1\) and therefore (iii) follows from (ii) of Lemma 2.

**References**


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