PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 98, Number 4, December 1986

A LATTICE-THEORETIC EQUIVALENT OF THE INVARIANT SUBSPACE PROBLEM

W. E. LONGSTAFF

ABSTRACT. Every bounded linear operator on complex infinite-dimensional separable Hilbert space has a proper invariant subspace if and only if for every lattice automorphism ϕ of a certain abstract complete lattice P (defined by N. Zierler) there exists an element $a \in P$ different from 0 and 1 such that $\phi^2(a) \leq a$.

Let H be a complex, infinite-dimensional, separable Hilbert space. The Invariant Subspace Problem is: does every operator $T \in B(H)$ have a nontrivial invariant subspace? Several equivalent problems are known; some are mentioned in [**3**] (e.g. p. 190, p. 194), see also [**2**]. In this note we point out that this famous problem is equivalent to a problem in lattice theory (Corollary 1).

Our equivalence rests on two results. Firstly, the lattice $\mathcal{C}(H)$ of all closed subspaces of H has a lattice-theoretic characterization due to Zierler [**4**, **5**]. Secondly, every lattice automorphism ϕ of $\mathcal{C}(H)$ is spatial in the sense that there exists a bicontinuous linear or conjugate linear bijection (unique up to nonzero scalar multiples) $S: H \to H$ such that $\phi(M) = SM$ for every $M \in \mathcal{C}(H)$, [**1**].

Let $\operatorname{Aut} \mathcal{C}(H)$ denote the group of automorphisms of $\mathcal{C}(H)$.

THEOREM 1. *The following are equivalent.*

(1) *For every $T \in B(H)$ there exists $M \in \mathcal{C}(H)$ different from* (0) *and H such that $TM \subseteq M$;*

(2) *For every $\phi \in \operatorname{Aut} \mathcal{C}(H)$ there exists $M \in \mathcal{C}(H)$ different from* (0) *and H such that $\phi^2(M) \subseteq M$.*

PROOF. Assume (1) holds. Let $\phi \in \operatorname{Aut} \mathcal{C}(H)$ be induced by S. Then ϕ^2 is induced by S^2 which is linear. Thus (2) holds.

Conversely, assume (2) holds. Let $T \in B(H)$ and let λ be a scalar satisfying $|\lambda| > \|T\|$. Then $S = T - \lambda$ is invertible and [**3**, p. 34] $S = R^2$ for some invertible operator $R \in B(H)$. If ϕ is the automorphism induced by R and M is as in (2) above, we have $SM \subseteq M$ so $TM \subseteq M$.

Let P be an abstract lattice satisfying the (lattice-theoretic) hypotheses of Theorem 2.2 of [**4**] and assume also that (in the notation of [**4**]) the coordinatizing division ring D is algebraically closed. Then there exists a lattice isomorphism $\theta: P \to \mathcal{C}(H)$ (satisfying $\theta(a') = \theta(a)^\perp$, where a' denotes the complement of a in P). Let $\operatorname{Aut} P$ denote the set of automorphisms of P, and let 0 (respectively, 1) denote the least (respectively, greatest) element of P.

Received by the editors July 8, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47A15; Secondary 06B05.

COROLLARY 1. *The following are equivalent.*

(1) *For every* $T \in B(H)$ *there exists* $M \in \mathcal{C}(H)$ *different from* (0) *and* H *such that* $TM \subseteq M$;

(2) *For every* $\phi \in \operatorname{Aut} P$ *there exists* $a \in P$ *different from* 0 *and* 1 *such that* $\phi^2(a) \leq a$.

Let $\mathcal{L}$ denote the set of automorphisms of $\mathcal{C}(H)$ that are induced by invertible operators of $B(H)$. Obviously $\mathcal{L}$ is a subgroup of Aut $\mathcal{C}(H)$. Some characterizations of $\mathcal{L}$ follow.

THEOREM 2. $\mathcal{L} = \{\phi^2\psi^2 \colon \phi, \psi \in \operatorname{Aut} \mathcal{C}(H)\}$.

PROOF. Let $\mathcal{Q} = \{\phi^2\psi^2 \colon \phi, \psi \in \operatorname{Aut} \mathcal{C}(H)\}$. Clearly $\mathcal{Q} \subseteq \mathcal{L}$. Let $\eta \in \mathcal{L}$ and suppose that η is induced by $S \in B(H)$. Then $S = UA$ with U unitary and A positive and invertible. There exists $V \in B(H)$ such that $V^2 = U$. Also $(A^{1/2})^2 = A$. Thus $\eta = \phi^2\psi^2$ where ϕ is induced by V and ψ is induced by $A^{1/2}$.

COROLLARY 2. $\mathcal{L}$ *is the subgroup of* $\operatorname{Aut} \mathcal{C}(H)$ *generated by* $\{\phi^2 \colon \phi \in \operatorname{Aut} \mathcal{C}(H)\}$.

If G is an abstract group, it is clear that a subgroup K of G has index 2 if and only if $gh \in K$ whenever $g, h \notin K$. Also, every subgroup of G of index 2 is a maximal proper (normal) subgroup and contains the square of every element of G. From these observations and the preceding corollary it follows that $\mathcal{L}$ is the only subgroup of Aut $\mathcal{C}(H)$ of index 2.

REFERENCES

1. P. A. Fillmore and W. E. Longstaff, *On isomorphisms of lattices of closed subspaces*, Canad. J. Math. **36** (1984), 820–829.
2. E. A. Nordgren, H. Radjavi, and Peter Rosenthal, *A geometric equivalent of the invariant subspace problem*, Proc. Amer. Math. Soc. **61** (1976), 66–68.
3. H. Radjavi and Peter Rosenthal, *Invariant subspaces*, Springer-Verlag, New York, 1973.
4. N. Zierler, *Axioms for nonrelativistic quantum mechanics*, Pacific J. Math. **11** (1961), 1151–1169.
5. ——, *On the lattice of closed subspaces of Hilbert space*, Pacific J. Math. **19** (1966), 583–586.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WESTERN AUSTRALIA, NEDLANDS 6009, AUSTRALIA