SOME NEW EXAMPLES
OF NONORIENTABLE MINIMAL SURFACES

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ABSTRACT. The classical Henneberg's minimal surface (1875, [3, 4, 11]) was
the unique nonorientable example known until 1981, when Meeks [6] exhibited
the first example of a nonorientable, regular, complete, minimal surface of finite
total curvature $-6\pi$.

In this paper, we study the nonorientable, regular, complete minimal surfaces
of finite total curvature and give some examples of punctured projective
planes regularly and minimally immersed in $\mathbb{R}^3$ and $\mathbb{R}^n$.

1. Nonorientable minimal surfaces in $\mathbb{R}^n$. We consider surfaces $S$ in $\mathbb{R}^n$
defined by maps $X: M \to \mathbb{R}^n$, where $M$ is a two-dimensional manifold.

For the study of a nonorientable, connected surface $S$ we take the double surface
$\tilde{S}$ given by $\tilde{X}: \tilde{M} \to \mathbb{R}^n$, where $\Pi: \tilde{M} \to M$ is the oriented two-sheeted covering of
$M$, and $\tilde{X} = X \circ \Pi$. We have an involution $I: \tilde{M} \to \tilde{M}$ without fixed points and a
conformal structure on $M$ such that $I$ is antiholomorphic.

We have a representation theorem for the nonorientable minimal surfaces:

THEOREM 1.1. Let $S$ be a nonorientable regular connected minimal surface in
$\mathbb{R}^n$. The double surface $\tilde{S}$ is a minimal surface in $\mathbb{R}^n$ defined by $\tilde{X}: \tilde{M} \to \mathbb{R}^n$,

$$\tilde{X}(p) = \text{Re} \int_{p_0}^{p} \tilde{\phi}(\zeta) \, d\zeta, \quad p_0, p \in \tilde{M},$$

with $\alpha = \tilde{\phi}(\zeta) \, d\zeta$ such that $I^{*}\alpha = \bar{\alpha}$ (that is, $I^{*}\alpha_k = \bar{\alpha}_k$, $1 \leq k \leq n$).

Conversely, if $\tilde{S}$ is a regular orientable connected minimal surface in $\mathbb{R}^n$ given by
(1.1) and if there is an antiholomorphic involution $I: \tilde{M} \to \tilde{M}$ without fixed points
such that $I^{*}\alpha = \bar{\alpha}$, then $\tilde{S}$ is the double surface of a regular nonorientable minimal
surface in $\mathbb{R}^n$.

The consequences of the condition $I^{*}\alpha = \bar{\alpha}$ with respect to the various forms of
representation for minimal surfaces in $\mathbb{R}^n$ are

COROLLARY 1.2 (MEEEKS [6]). Let $f$ and $g$ be the functions of Weierstrass's
representation of an orientable, regular, connected minimal surface $\tilde{S}$ in $\mathbb{R}^3$, that
is

$$\tilde{X}(p) = \text{Re} \int_{p_0}^{p} \frac{f(\zeta)}{2} (1 - g^2(\zeta), i(1 + g^2(\zeta)), 2g(\zeta)) \, d\zeta, \quad p_0, p \in \tilde{M}.$$

The surface $\tilde{S}$ is the double surface of a nonorientable, regular minimal surface in
$\mathbb{R}^3$ if and only if

$$\left\{ \begin{align*}
(i) \quad & g(I(p)) = -1/g(p), \quad p \in \tilde{M}, \\
(ii) \quad & I^{*}(\omega) = -g^2\omega, \quad \omega = f(z) \, dz,
\end{align*} \right.$$  

for some antiholomorphic involution $I: \tilde{M} \to \tilde{M}$ without fixed points.
COROLLARY 1.3. Let \( \tilde{S} \) be an orientable, regular, connected minimal surface in \( \mathbb{R}^4 \) given by \( \tilde{X}: \tilde{M} \to \mathbb{R}^4 \),

\[
\tilde{X}(p) = \text{Re} \int_{p_0}^p (1 + g_1 g_2, i(1 - g_1 g_2), g_1 - g_2, -i(g_1 + g_2)) \omega, \quad p_0, p \in \tilde{M}.
\]

Then, \( \tilde{S} \) is the double surface of a nonorientable minimal surface in \( \mathbb{R}^4 \) if and only if, for some antiholomorphic involution \( \tilde{I} \) on \( \tilde{M} \) without fixed points,

\[
\begin{align*}
(1.3) \quad & g_k(\tilde{I}(p)) = -\frac{1}{g_k(p)}, \quad k = 1, 2, \text{ and} \\
& \tilde{I}^* \omega = \overline{g_1 g_2 \omega}, \quad \omega = f(z) \, dz,
\end{align*}
\]

If \( C(\tilde{S}) \) is the total curvature of \( \tilde{S} \), we can define the total curvature of \( S \) by

\[(1.4) \quad C(S) = C(\tilde{S})/2.\]

In what follows, the double surface \( \tilde{S} \) will always be connected, complete and of finite total curvature \( C(\tilde{S}) = 2\pi \tilde{m} \). From Chern and Osserman's theorem [1] we have \( \tilde{M} \) conformally equivalent to a compact Riemann surface of genus \( \tilde{\gamma} \) punctured at \( \tilde{r} = 2r \) points \( \{p_1, p_2, \ldots, p_r, q_1, \ldots, q_r\} \), with \( \tilde{I}(p_j) = q_j, \tilde{I}(q_j) \) being the extension of \( \tilde{I} \) to \( p_j \), \( 1 \leq j \leq r \). The function \( \tilde{\phi} \) of (1.1) has poles of order \( m_j \) at \( p_j \) and \( q_j \), and

\[(1.5) \quad \tilde{m} - 2 \sum_{j=1}^r m_j = 2\tilde{\gamma} - 2.\]

The Chern and Osserman inequality for nonorientable minimal surfaces is

\[(1.6) \quad C(S) \leq 2\pi (\chi(M) - r),\]

with \( \chi(M) \) the Euler characteristic of \( M \).

A nonorientable version of Gackstatter's theorem [2] gives an estimate for the dimension of a nonorientable, complete, minimal surface \( S \) with total curvature \( C(S) = -2\pi m \), \( r \) ends and genus \( \gamma \):

\[(1.7) \quad \text{Dim}(S) \leq 2m - 2\gamma - r + 3.\]

We have

PROPOSITION 1.4. The total curvature of a nonorientable, regular, complete minimal surface is at most \(-4\pi\).

PROOF. From (1.7) and the fact that \( \text{Dim} S \geq 3, 2m \geq 4 \).

2. Genus one nonorientable complete minimal surfaces in \( \mathbb{R}^n \). A characterization of the complete minimal surfaces of finite total curvature and genus
one can be set up combining Theorem 1.1 and Hoffman and Osserman's theorem [5]:

THEOREM 2.1. Let \( \tilde{M} \) be the complex plane minus \( 2r-1 \) points \( \{0, z_1, \ldots, z_{r-1}, -1/z_1, \ldots, -1/z_{r-1}\} \). Let the complex vector \( \phi(\zeta) \) be of the form

\[
\phi(\zeta) = F(\zeta)(p_1(\zeta), \ldots, p_n(\zeta)),
\]

with

\[
F(\zeta) = \left( \zeta^{-r} \prod_{k=1}^{r-1} (\zeta - z_k)^{\nu_k}(\zeta + 1/z_k)^{\nu_k} \right)^{-1},
\]

\( p_j(\zeta) \) satisfying

(i) \( p_j(\zeta) \) is a polynomial,
(ii) the maximum degree of \( p_j \) is \( 2m \),
(iii) the \( p_j \)'s have no common factor,
(iv) \( \sum_{j=1}^{n} p_j^2 \equiv 0, \)
(v) \( (-1)^{m+1}z^{2m}p_j(-1/\zeta) = p_j(\zeta), 1 \leq j \leq m. \)

Suppose that the \( \nu_k \)'s satisfy \( \sum_{k=0}^{r-1} \nu_k = m + 1, \nu_k \geq 2, 0 \leq k \leq r - 1 \) and \( \text{Re} \int_{\gamma} \phi = 0, \) for any simple closed curve \( \gamma \) in \( \tilde{M} \). Then \( \tilde{X}(\zeta) = \text{Re} \int_{p_0}^{p} \phi(\zeta) d\zeta \) defines a double minimal surface associated to a nonorientable regular complete minimal surface in \( \mathbb{R}^n \) of genus one, \( r \) ends and total curvature \( -2\pi m. \)

Conversely, every genus one nonorientable minimal surface in \( \mathbb{R}^n \) admits a representation as mentioned above.

PROOF. The double surface associated to a genus one nonorientable minimal surface is conformally equivalent to the complex plane \( \mathbb{C} \) minus \( 2r \) points with the involution \( \tilde{I}: \mathbb{C} \to \mathbb{C} \) given by \( \tilde{I}(z) = -1/z \), that is,

\[
\tilde{M} = \mathbb{C} - \{0, z_1, \ldots, z_{r-1}, -1/z_1, \ldots, -1/z_{r-1}\}.
\]

From Hoffman and Osserman's result, \( \phi = F(p_1, \ldots, p_n), \) \( p_j \) polynomials satisfying (i)–(iv).

The map \( \tilde{X}: \tilde{M} \to \mathbb{R}^n \) defines a double surface if and only if \( \tilde{I}(\phi_j(\zeta) d\zeta) = \phi_j(\zeta) d\zeta, 1 \leq j \leq n; \) from this fact and (1.5), we have \( \sum_{k=0}^{r-1} \nu_k = m + 1 \) and (v). \( \square \)

We study initially surfaces whose total curvature is the upper bound given by Proposition 1.4.

THEOREM 2.2. Let \( S \) be a nonorientable, regular, complete minimal surface of finite total curvature \( -4\pi \). Then \( S \) is a minimal immersion of the projective plane minus one point and lies fully in \( \mathbb{R}^4 \); any two such minimal surfaces are similar.

PROOF. From (1.5), and (1.6) and (1.7), using \( \chi(M) = 2 - \gamma - r, \) we have \( \gamma = 1, r = 1 \) and \( m_1 = 3. \) The double surface has genus zero and two ends; thus, by Theorem 2.1, it can be given by \( \tilde{X}: \mathbb{C} - \{0\} \to \mathbb{R}^n, \)

\[
\tilde{X}_j(p) = \text{Re} \int_{p_0}^{p} \phi_j(\zeta) d\zeta, \quad p_0, p \in \tilde{M} = \mathbb{C} - \{0\}, \phi_j(\zeta) = p_j(\zeta)/\zeta^3,
\]

\( p_j \) satisfying conditions (i)–(v).
Setting \( p_j(\zeta) = a_j \zeta^4 + b_j \zeta^3 + c_j \zeta^2 + d_j \zeta + e_j \), we can verify that \( a_j = -e_j \), \( b_j = d_j \), \( c_j = 0 \) and that \( A_1 = (\text{Re} a_1, \ldots, \text{Re} a_n) \), \( A_2 = (\text{Im} a_1, \ldots, \text{Im} a_n) \), \( B_1 = (\text{Re} b_1, \ldots, \text{Re} b_n) \) and \( B_2 = (\text{Im} b_1, \ldots, \text{Im} b_n) \) are real orthogonal vectors of the same length.

Thus, \( A = (1, i, 0, 0) \) and \( B = (0, 0, 1, i) \) give a solution in \( \mathbb{R}^4 \); two solutions \( \tilde{X} \) and \( \tilde{Y} \) are such that \( \tilde{Y} = \lambda M \tilde{X} + X_0 \), with \( \lambda \in \mathbb{R} \) and \( M \) a real orthogonal matrix.

Immediately, we have Meeks’ theorem:

**COROLLARY 2.3.** The total curvature of a nonorientable, regular, complete minimal surface in \( \mathbb{R}^3 \) is at most \(-6\pi\).

The total curvature of an orientable minimal surface in \( \mathbb{R}^3 \) is of the form \( C(S) = -4\pi \cdot \text{degree}(g) \), with \( g \) the meromorphic function of the Weierstrass representation. Thus, for nonorientable minimal surfaces,

\[
C(S) = -2\pi \cdot \text{degree}(g),
\]

with \( g: \tilde{M} \to \mathbb{C} \) the function of the Weierstrass representation of the double surface \( \tilde{S} \), which is a rational function when it has finite degree and the genus of \( \tilde{M} \) is zero.

The next result is very important for the construction of examples:

**PROPOSITION 2.4.** Let \( g \) be a rational complex function satisfying \( g(-1/z) = -1/g(z) \). Then, the degree of \( g \) is odd and \( g \) is of the form

\[
g(z) = cz^m \prod_{j=1}^{m}(z - a_j) \prod_{j=1}^{m}(z + 1/a_j), \quad \text{with } |c| \prod_{j=1}^{m}|a_j| = 1.
\]

**PROOF.** Let \( g \) be a rational function, and let

\[
g(z) = cz^m \prod_{j=1}^{m}(z - a_j) \prod_{k=1}^{m}(z - b_k), \quad a_j \neq b_k \text{ for any } j, k, a_j \neq 0, b_k \neq 0,
\]

where the factors appear with multiplicities. From the condition

\[
g(-1/z) = -1/g(z),
\]

with some calculations, we complete the proof. \( \square \)

**PROPOSITION 2.5.** The total curvature of a nonorientable regular complete minimal surface in \( \mathbb{R}^3 \) of genus one is of the form \( C(S) = -2\pi m, m \text{ odd, } m \geq 3 \).

**PROOF.** This follows immediately from Proposition 2.4. \( \square \)

Meeks [6] has proved that there exists a unique nonorientable regular complete minimal surface in \( \mathbb{R}^3 \) of total curvature \(-6\pi \). We extend Meeks’ surface in the sense of

**THEOREM 2.6.** There exists a nonorientable regular complete minimal surface in \( \mathbb{R}^3 \) of genus one, one end and total curvature \(-2\pi m \), for any \( m \) odd, \( m \geq 3 \).

**PROOF.** With straightforward calculations, one can show that

\[
f(z) = i(z + 1)^2/z^{m+1} \quad \text{and} \quad g(z) = z^{m-1}(z - 1)/(z + 1)
\]

are the functions of the Weierstrass representation of a double minimal surface \( \tilde{S} \); that is, the map

\[
\tilde{X}(p) = \text{Re} \int_{p_0}^{p} \frac{f}{2}(1 - g^2, i(1 + g^2), 2g) \, d\zeta, \quad p_0, p \in \mathbb{C} - \{0\},
\]

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satisfies (i)–(v) of Theorem 2.1. The surfaces obtained are regular and complete. □

REMARKS. 1. If \( n = 3 \) the surface of Theorem 2.6 is exactly Meeks' surface.
2. The surface is an infinite Moebius band with \((m - 1)/2\) twists and the image of \(|z| = 1\) is a circle centered at \((0, -2/(m - 1), 0)\) with radius \(2/(m - 1)\) covered \((m - 1)\) times.
3. The normals to the surface along \(|z| = 1\) have the third component \(N_3|z|=1 = \text{Re}(z)|z|=1\).

The Chern-Osserman inequality for nonorientable minimal surfaces of total curvature \(-6\pi\) gives us the relation \(\gamma + 2r \leq 5\); hence, if \(\gamma = 1\), we may have \(r = 1\) or \(r = 2\). Meeks has proved that the case \(r = 2, C = -6\pi\) cannot occur. Thus, we have

**PROPOSITION 2.7.** The total curvature of a nonorientable regular complete minimal surface in \(\mathbb{R}^3\) of genus one and two ends is at most \(-10\pi\).

For \(C = -10\pi\), we have

**THEOREM 2.8.** There exists a nonorientable regular complete minimal surface in \(\mathbb{R}^3\) of genus one, two ends and total curvature \(-10\pi\).

**PROOF.** The double surface associated to such a surface will be an orientable surface of genus zero, four ends with total curvature \(-20\pi\) and, by Theorem 2.1, it may be given by

\[
\tilde{X}: C - \{0, 1, -1\} \to \mathbb{R}^3, \quad \tilde{I}: C \to C, \quad \tilde{I}(z) = -1/z.
\]

Let \(f(z) = a(b^2z^2 - 1)^2/z^2(z - 1)^4(z + 1)^4\) and \(g(z) = ab^2(z^2 - b^2)/(\tilde{b}^2z^2 - 1)\); \(f\) and \(g\) are the functions of the Weierstrass representation of a double minimal surface if and only if

\[
|\alpha| |b|^2 = 1 \quad \text{and} \quad a = -\alpha\alpha b^4 \quad \text{or} \quad aab^2 = -\alpha\alpha b^2.
\]

The completeness and regularity can easily be verified. To choose \(\alpha, a\) and \(b\) such that

\[
\text{Re} \int_\gamma \frac{f(z)}{2} (1 - g^2(z), i(1 + g^2(z)), 2g(z)) \, dz = 0,
\]

for any closed curve \(\gamma\) in \(C - \{0, 1, -1\}\), we observe that, by Cauchy's integral formula, for simple closed curves \(\gamma_0\) and \(\gamma_1\) around \(z = 0\) and \(z = 1\), respectively,

\[
\int_{\gamma_0} \frac{p_j(z) \, dz}{z^2(z - 1)^4(z + 1)^4} = 2\pi i p_j'(0),
\]

and

\[
\int_{\gamma_1} \frac{p_j(z) \, dz}{z^2(z - 1)^4(z + 1)^4} = \frac{2\pi i}{3!} \cdot \frac{1}{16} [p_j''(1) - 12p_j'(1) + 57p_j''(1) - 105p_j'(1)].
\]

Calling \(R_j = p_j''(1) - 12p_j'(1) + 57p_j'(1) - 105p_j'(1), j = 1, 2, 3\), we have

\[
R_1 = -105(a - \bar{a}) + 30(ab^2 - \bar{a}b^2) + 3(ab^4 - \bar{a}b^4),
\]

\[
R_2 = i[-105(a + \bar{a}) + 30(ab^2 + \bar{a}b^2) + 3(ab^4 + \bar{a}b^4)],
\]

\[
R_3 = 0.
\]
Thus, \( \text{Re}[2\pi i R_j] = 0, j = 1, 2 \) if \(-105a + 30ab^2 + 3ab^4 = 0\).

For each \( b, \alpha = -1/b^2 \), \( a = i \) satisfy (2.2) and

\[
\tilde{X}(p) = \text{Re} \int_{p_0}^p \frac{f}{2}(1 - g^2, i(1 + g^2), 2g) \, d\zeta
\]
gives the double surface. \( \square \)

We investigate the problem of the regular minimal surfaces in \( \mathbb{R}^3 \) of finite total curvature, genus one and three ends, and we have

**Theorem 2.9.** There exists a nonorientable regular complete minimal surface in \( \mathbb{R}^3 \) of genus one, three ends and total curvature \(-14\pi\).

**Proof.** We construct the double surface \( \tilde{S} \) defined by \( \tilde{X} : \mathbb{C} \setminus \{0, 1, -1, i, -i\} \to \mathbb{R}^3 \) with the functions \( f \) and \( g \) of the Weierstrass representation (1.2) given by

\[
g(z) = \frac{ab^4 z^3(z^4 - b^4)/(b^4 z^4 - 1)} \quad f(z) = \frac{a(\bar{b} z^4 - 1)^2/z^2(\bar{z}^2 - 1)^3(z^2 + 1)^3}.
\]

The functions \( f \) and \( g \) satisfy (1.2) if and only if

\[
|\alpha| |b|^4 = 1, \quad \alpha = \frac{a}{\bar{a}a^2 b^8}.
\]

With some calculations, we can verify that the real parts of the integrals

\[
\int_{\gamma} \frac{p_j(z) \, dz}{z^2(z^4 - 1)^3},
\]

where \( \gamma \) is a simple closed curve, are null for any \( j, j = 1, 2, 3 \), provided \( b \) satisfies the following condition: \( 3b^8 + 1064 - 45 = 0 \). Thus, \( b^4 \in \mathbb{R} \).

Choosing \( a = 1/b^4 \) and \( \alpha = 1 \) the conditions (2.3) are verified. The completeness and regularity can be easily checked. \( \square \)

In \( \mathbb{R}^4 \), exploring the properties of the functions \( g_1, g_2 \) of the representation (1.3), we have

**Theorem 2.10.** For any integer \( m, m \geq 2 \), there exists a nonorientable regular complete minimal surface \( S \) in \( \mathbb{R}^4 \) of genus one, one end and total curvature \(-2\pi m\).

**Proof.** The double surface \( \tilde{S} \) associated to \( S \) has total curvature \(-4\pi m = -2\pi \tilde{m}, \tilde{m} = 2m \), genus zero and two ends. We set the functions \( f, g_1, g_2 \) in the representation (1.3) by \( f(z) = z^{m-1}, g_1(z) = \varepsilon_1/z^{k_1}, g_2(z) = \varepsilon_2/z^{k_2}, \) with \( \tilde{m} = 2m = k_1 + k_2, k_1, k_2 \) odd, \( k_1, k_2 \geq 1 \), and \( \tilde{M} = \mathbb{C} \setminus \{0\} \).

The functions \( f, g_1 \) and \( g_2 \) satisfy (i) and (ii) of Corollary 1.3, if and only if

\[
|\varepsilon_1| = |\varepsilon_2| = 1, \quad \varepsilon_1 \varepsilon_2 = (-1)^{m-1}.
\]

To verify that \( \text{Re} \int_{\gamma} \phi(\zeta) \, d\zeta = 0 \), for any closed curve in \( \mathbb{C} \setminus \{0\} \), we observe that

\[
\phi_1(\zeta) = f(1 + g_1 g_2) = z^{m-1} + \varepsilon_1 \varepsilon_2 / z^{m+1},
\]

\[
\phi_2(\zeta) = i f(1 - g_1 g_2) = i(z^{m-1} - \varepsilon_1 \varepsilon_2 / z^{m+1}),
\]

\[
\phi_3(\zeta) = f(g_1 - g_2) = \varepsilon_1 z^{k_2} - \varepsilon_2 z^{k_1} / z^{m+1},
\]

\[
\phi_4(\zeta) = -i f(g_1 + g_2) = -i(\varepsilon_1 z^{k_2} + \varepsilon_2 z^{k_1}) / z^{m+1},
\]
and rewriting $k_1 = m - n$, $k_2 = m + n$, $n \geq 1$, we have $\int_\gamma \phi_j = 0$, $j = 1, \ldots, 4$. Thus, the double surface $\tilde{S}$ is defined by

$$\tilde{X}(p) = \text{Re} \int_{p_0}^{p} \phi(\zeta) \, d\zeta, \quad p, p_0 \in \mathbb{C} - \{0\}. \quad \Box$$

For $\gamma = 1, r = 1$, the dimension of $S$ satisfies $\text{Dim}(S) \leq 2m$. We will show that this upper bound is sharp.

**Theorem 2.11.** For any integer $m$, $m \geq 2$, there exists a nonorientable regular complete minimal surface $S$ in $\mathbb{R}^{2m}$ of genus one, one end and total curvature $-2\pi m$ that lies fully in $\mathbb{R}^{2m}$.

**Proof.** The double surface $\tilde{S}$ associated to $S$, by Theorem 2.1, is given by $\tilde{X}: \mathbb{C} - \{0\} \to \mathbb{R}^n$, $\tilde{X}(p) = \text{Re} \int_{p_0}^{p} \phi(\zeta) \, d\zeta$, with $\phi_j(\zeta) = p_j(\zeta)/\zeta^{m+1}$, $p_j(\zeta)$ satisfying (i)-(v).

Particularly, condition (v) is equivalent to

$$(2.5) \quad (-1)^{m+1} \zeta^{-2m} p_j(-1/\zeta) = \overline{p_j(\zeta)}, \quad 1 \leq j \leq n.$$  

Taking $p_j(\zeta) = a_0^j + a_1^j \zeta + \cdots + a_{2m-1}^j \zeta^{2m-1} + a_{2m}^j \zeta^m$ in (2.5) we obtain

$$(-1)^{m+1} (a_0^j \zeta^{-2m} - a_1^j \zeta^{-2m-1} + \cdots + a_{2m}^j) = (a_0^{j-2m} + a_1^{j-2m-1} + \cdots + a_{2m}^j).$$

Then, there are two possibilities:

(a) $(m + 1)$ even: $a_0^j = \overline{a_{2m}^j}$, $a_1^j = \overline{a_{2m-1}^j}$, $\ldots$, $a_{m-1}^j = \overline{a_m^j}$, $a_m^j = \overline{a_0^j}$,

(b) $(m + 1)$ odd: $a_0^j = -\overline{a_{2m}^j}$, $a_1^j = \overline{a_{2m-1}^j}$, $\ldots$, $a_{m-1}^j = \overline{a_m^j}$, $a_m^j = -\overline{a_0^j}$.

The integrals $\int_\gamma \phi_j(\zeta) \, d\zeta$, $\gamma$ a closed curve in $\mathbb{C} - \{0\}$, should have null real part; therefore, $\text{Re}(2\pi i \cdot a_m^j) = 0$, $1 \leq j \leq n$, that is $\text{Im} a_m^j = 0$, $1 \leq j \leq n$. Comparing with the last condition in (a) or (b), we have $a_m^j = 0$, $1 \leq j \leq n$.

To determine solutions in $\mathbb{R}^n$, we should find complex vectors $A_k = (a_0^k, \ldots, a_n^k)$, $0 \leq k \leq m - 1$, such that the corresponding polynomials satisfy $\sum_{j=1}^n p_j^k(\zeta) \equiv 0$.

It is easy to verify that $A_0 = (1, i, 0, \ldots, 0)$, $A_1 = (0, 0, 1, i, 0, \ldots, 0)$, $A_{m-1} = (0, \ldots, 0, 1, i)$ give a solution in $\mathbb{C}^m$ if $m$ is even and $A_0 = (1, i, 0, \ldots, 0)$, $A_2 = (0, 0, 1, i, 0, \ldots, 0)$, $\ldots$, $A_{m-2} = (0, 0, \ldots, 0, \sqrt{2}, i\sqrt{2}, 0, 0)$, $A_{m-1} = (0, 0, \ldots, 0, 1, i)$ give a solution if $m$ is odd. \quad \Box

With these results, we have partial answers for the problem of existence of nonorientable, regular, minimal surfaces in $\mathbb{R}^3$ and $\mathbb{R}^n$ posed by Nitsche [7, 1965]. We do not know, for instance, an example of a genus two nonorientable minimal surface in $\mathbb{R}^3$, that is, a minimal Klein bottle in $\mathbb{R}^3$.

**References**

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