MINIMAL DEGREES OF FAITHFUL CHARACTERS
OF FINITE GROUPS WITH A T.I. SYLOW $p$-SUBGROUP

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Abstract. Using the classification of the finite simple groups we show in this article that a faithful complex character $\chi$ of a finite group $G$ with a nonnormal T.I. Sylow $p$-subgroup $P$ has degree $\chi(1) > \sqrt{|P|} - 1$. This result verifies a conjecture of H. S. Leonard [10].

Introduction. Let $p$ be a fixed prime, and let $G$ be a finite group with a T.I. Sylow $p$-subgroup $P$. That is, two different conjugates of $P$ have only the identity element in common. In [10] H. S. Leonard conjectured that if $G$ has a faithful complex character $\chi$ with degree $\chi(1) \leq \sqrt{|P|} - 1$, then $P$ is normal in $G$. Using the classification of the finite simple groups we prove Leonard’s conjecture in this note (Theorem 3.2).

In §1 this theorem is first proved for $p$-solvable groups $G$ (Proposition 1.3). Then we determine the composition series of a minimal counterexample $G$ to Leonard’s conjecture (Proposition 1.4). Since by Sibley’s theorem [12] the main result of this article is known if $P$ is cyclic, we give in §2 a complete list of all finite simple groups $G$ having a noncyclic T.I. Sylow $p$-subgroup for some prime $p$ (Proposition 2.3). Here for odd $p$ we use Gorenstein and Lyons’ theorem [4] classifying all finite groups $G$ with $O_p'(G) = 1$, $p$-rank $m_p(G) > 1$, and containing a strongly $p$-embedded subgroup. If $p = 2$, then Proposition 2.3 is only a restatement of Suzuki’s theorem [13]. After these preparations Leonard’s conjecture is proved in §3. In Remark 3.3 we show that the bound of Theorem 3.2 cannot be replaced by $\frac{1}{2}(|P| - 1)$, which is the bound of Sibley’s theorem [12].

For notation and terminology we refer to the books by Feit [1], Gorenstein [2, 3], Huppert [5], Huppert and Blackburn [6], and Landrock [9]. All character tables of finite simple groups used here are contained in the CAS-system [11] of J. Neubüser, H. Pahlings, and W. Plesken (TH. Aachen, Federal Republic of Germany).

1. Reduction to almost simple groups. In this section we determine the structure of a finite group $G$ of minimal order among the groups $H$ without a normal Sylow $p$-subgroup, but satisfying the hypothesis of Leonard’s conjecture.

The following lemma due to Feit [1, p. 123] is our basic tool.

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Lemma 1.1. Let $S$ be a splitting field of characteristic zero for the finite group $G$ with a T.I. Sylow $p$-subgroup $P$. Let $\chi$ be a character of $SG$ such that $\chi(1)^2 \leq |P|$. Let $H$ be a subgroup of $G$ containing $N_G(P)$. Then $(\chi, \chi)_G = (\chi, \chi)_H$.

For a short proof of this result we refer to [9, p. 129].

Lemma 1.2. Let $G$ be a finite group with a T.I. Sylow $p$-subgroup $P$. Then the following assertions hold.

(a) Every subgroup $U$ of $G$ with $p \mid |U|$ has a T.I. Sylow $p$-subgroup.
(b) $G/N$ has a T.I. Sylow $p$-subgroup for every normal subgroup $N$ of $G$ with $(p, |N|) = 1$.
(c) $C_G(x)$ has a normal Sylow $p$-subgroup for every $1 \neq x \in P$.

Proof. See Suzuki [13, p. 59].

Proposition 1.3. Let $G$ be a $p$-solvable group with T.I. Sylow $p$-subgroup $P$ and a faithful complex character $\chi$ such that $\chi(1)^2 \leq |P|$. Then $P$ is a normal subgroup of $G$.

Proof. Let $G$ be a minimal counterexample and let $(F = R/\pi, R, S = \text{quot}(R))$ be a splitting $p$-modular system for $G$ (see [9, p. 47]). Then $O_p(G) = 1$, and so $Q = O_p(P) \neq 1$. Let $H = QN_G(P)$. If $H \neq G$, then $H$ by induction. Hence $O_{p^*}(H) = O_{p^*}(H) \times P \geq O_{p^*}(P) > Q$.

because $G$ is $p$-solvable. This forces $O_p(G) \neq 1$, a contradiction. Therefore, $G = H$. But $G = O_{p^*}(G)$ by minimality, whence $G = QP$. In particular, $G$ is $p$-nilpotent, and every $p$-block $B$ of $G$ contains only one modular character by Theorem 14.9 of [6]. Since $P$ is a T.I. Sylow $p$-subgroup, and $\chi(1) < |P|$ it follows from Theorem 14.8 of [6] that (the possibly reducible) $\chi$ contains an irreducible constituent $\mu$ belonging to a nonprincipal $p$-block $B$ with defect group $\delta(B) = G_P$, because $\chi$ is faithful. As $G$ is $p$-nilpotent, by Theorem 2.1 of [1, p. 419], we also may assume that $\mu$ remains irreducible under restriction modulo $\pi$. Let $\bar{\mu}$ be a module over $F$ affording $\mu$ modulo $\pi$.

Let $b$ be the block of $U = N_G(P)$ associated with $B$ by the Brauer correspondence. Since $\mu(1)^2 < |P|$, Lemma 1.1 asserts that $\mu | U$ is irreducible. Because $P$ is a T.I. set, Green's correspondence theorem implies that $\bar{\mu} | U$ is an indecomposable module in $b$. Notice that $b$ contains only one modular irreducible character, so that all composition factors of $\bar{\mu} | U$ are isomorphic. In particular, if $\bar{\mu} | U$ is not irreducible, it has a nonzero nilpotent $FU$-endomorphism $\tau$, namely any nonzero map from the head to the socle of $\bar{\mu} | U$. As $P$ is a T.I. set, Corollary 5.8 of [9, p. 122], implies that $\tau$ is a projective endomorphism of $\bar{\mu} | U$. Therefore, $\mu(1)^2 > |P|$ by Corollary 6.11 of [9, p. 128], a contradiction. We now know that $\bar{\mu} | U$ is an irreducible $FU$-module of $b$ and that $P$ acts trivially on $\bar{\mu} | U$. Let $A = \ker \bar{\mu}$ in $G$. Now $A \neq G$ since $B$ is not the principal block. We have

$$P \leq \ker \bar{\mu} = A < G.$$ 

Therefore, $Q \nleq A$ as $G = QP$.

By Lemma 1.2 $P$ is a T.I. Sylow $p$-subgroup of $A$, and $\chi_A$ is a faithful complex
character of $A$ with $\chi_A(1)^2 < |P|$. So $P \triangleleft A$ by induction, which implies $P \triangleleft G$. This contradiction completes the proof.

The center of the group $G$ is denoted by $Z(G)$.

**Proposition 1.4.** Let $G$ be a minimal counterexample to Leonard’s conjecture. Let $P$ be a T.I. Sylow $p$-subgroup of $G$, and let $\chi$ be a faithful complex character of degree $\chi(1) \leq \sqrt{|P|} - 1$. Then:

(a) $Z(G) = O_p'(G) / O_p'(G) = H = O_p'(G)$.

(b) $H / Z(G)$ is a nonabelian simple group with a T.I. Sylow $p$-subgroup.

(c) $\chi$ may be assumed to be irreducible.

**Proof.** As $G$ is a minimal counterexample, $G = O_p'(G)$. Let $H = O_p'(G)$, and let $H / N \cong 1$ be a chief factor of $G$.

Suppose that $H / N$ is a $p'$-group. By Proposition 1.3 $G$ is not $p$-solvable. Thus $P_0 = P \cap N \neq 1$, and $L = N_G(P_0) < G$. Since $P \cap N \triangleleft P$, $P \triangleleft L$. So by induction $P \triangleleft L$. As $G = O_p'(G)$, and as $G = NL$ by the Frattini argument, we obtain $G = NP$, and so $N = H$, a contradiction.

Therefore $H / N$ is a direct product of isomorphic nonabelian simple groups $A$ with $p^{|A|}$. Hence $NN_G(P) < G$, which implies $P \triangleleft NN_G(P)$ by induction. Since $P$ is a T.I. set in $G$, we now get $O_p(N) = N \cap P = 1$. So $N$ is a $p'$-group commuting with $P$. Hence $C_G(N) \supset \langle P^g | g \in G \rangle = O_p'(G) = G$,

and so $Z(G) = N \leq O_p'(G)$.

As $(p, |A|) = 1$, Lemma 1.2 asserts that $H / N$ has a T.I. Sylow $p$-subgroup. Thus by Lemma 1.2(c) $H / N$ is simple. Since $O_p'(G) \subset O_p'(G) = H$, it follows that $N = O_p'(G)$.

Finally, we may replace $\chi$ by an irreducible constituent, which does not have $H$ in its kernel. This completes the proof.

2. **Simple groups with a noncyclic T.I. Sylow $p$-subgroup.** In this section we list the simple groups with a noncyclic T.I. Sylow $p$-subgroup. In [13] Suzuki classified the simple groups with such a Sylow $2$-subgroup. For odd primes $p$ our subsidiary result follows from Gorenstein and Lyons’ classification [4] of the finite groups $G$ with $O_p(G) = 1$, $p$-rank $m_p(G) > 1$, and containing a strongly $p$-embedded subgroup.

Here $m_p(G)$ denotes the maximum rank of an elementary abelian subgroup of a Sylow $p$-subgroup $P$ of the finite group $G$.

**Definition [3].** Let $P$ be a Sylow $p$-subgroup of the finite group $G$, and let $k$ be a positive integer. The $k$-generated $p$-core of $G$ is $\Gamma_{P,k}(G) = \langle N_G(Q) | Q \leq P, m_p(Q) \geq k \rangle$.

The proper subgroup $M$ of $G$ is called strongly $p$-embedded in $G$ if $\Gamma_{P,1}(G) \leq M$.

**Remark 2.1.** If the finite group $G$ contains a nonnormal T.I. Sylow $p$-subgroup $P$, then $M = N_G(P)$ is strongly $p$-embedded in $G$, as is easily seen.

A finite group $G$ is quasi-simple if $G = G'$ and $G / Z(G)$ is simple. The layer $L(G)$ of $G$ is the product of all subnormal quasi-simple subgroups of $G$, where $L(G) = 1$ if no such subnormal subgroup exists. The generalized Fitting subgroup of the finite group $G$ is defined as $F^*(G) = F(G)L(G)$, where $F(G)$ denotes the Fitting subgroup of $G$ (see [3, p. 44]).
In view of the classification theorem of the finite simple groups we now can restate Theorems (24.1), (24.2), and (24.9) of Gorenstein and Lyons [4, pp. 307, 311, and 318, respectively], as

**Proposition 2.2.** Let $p$ be an odd prime, $M$ a strongly $p$-embedded subgroup of the finite group $G$ with $O_p(G) = 1$ and $m_p(G) > 1$. Let $V = O^*(G)$ and let $P$ be a Sylow $p$-subgroup of $M$. Then $F^*(G) = L(V)$ is simple and one of the following holds.

1. $V \cong \text{PSL}_2(p^n)$ or $\text{PSU}_3(p^n)$, and $M = N_G(P)$.
2. $V \cong \mathbb{A}_p$ and $F^*(M) \cong \mathbb{A}_p \times \mathbb{A}_p$.
3. $p = 3$, $V \cong G_2(3^{2m+1})$, and $M = N_G(P)$, where $m \geq 0$.
4. $p = 3$, $V \cong M_{11}$ or $\text{PSL}_3(4)$, and $M = N_G(P)$.
5. $p = 5$, $V \cong M_{22}$, and $V \cap M \cong \text{Aut}(D_4(2))$.
6. $p = 5$, $V \cong 2F_4(2)', \text{Aut}(2B_2(2^5))$ or $M_c$, and $M = N_G(P)$.
7. $p = 11$, $V \cong J_4$, and $M = N_G(P)$.

**Proof.** By hypothesis, $\Gamma_{p,1}(G) \leq M \neq G$ and $P \leq V \cap M$. Thus $O_p(G) = 1 = F(G)$, because otherwise $G = N_G(O_p(G)) \leq \Gamma_{p,1}(G) \leq M \neq G$, a contradiction.

Let $K$ be a normal subgroup of $G$. As $O_p(G) = 1$, $P_0 = P \cap K \neq 1$. The Frattini argument asserts that $G = N_G(P_0)K$. Hence $K \neq \Gamma_{p,1}(G)$. It follows that every quasi-simple subnormal subgroup $L$ of $G$ is simple and $L \neq \Gamma_{p,1}(L)$, where $P_1 \in \text{Syl}_p(L)$.

Thus $F^*(G) = L(G) = L(V)$ is a direct product of simple groups $E_i$, $1 \leq i \leq k$, each of which contains a strongly $p$-embedded subgroup.

Let $E \in \{E_i|1 \leq i \leq k\}$, $P^* = P \cap L(V)$, and $X = EP^*$. Then $O_p(X) = 1 = O_p(K)$ and $\Gamma_{p,1}(X) \neq X$, because $P^* \subset \Gamma_{p,1}(G)$, but $X \neq \Gamma_{p,1}(G)$. Applying now Theorem (24.9)(4) of Gorenstein and Lyons [4, p. 318], we obtain that $\Omega_1(P^*) \leq E$ or $E \in \{G_2(3^r), 2B_2(2^5)\}$. Since $P^* \in \text{Syl}_p(L(V))$ it follows that $P^* \leq E$. As $O_p(L(V)) = 1$ we get $F^*(G) = L(G) = L(V) = E$. Hence $F^*(G)$ is simple. Now Theorems (24.1) and (24.2) of Gorenstein and Lyons [4, pp. 307, 311] complete the proof.

Combining this result with Suzuki's theorem [13] we obtain

**Proposition 2.3.** Let $G$ be a nonabelian simple group with a noncyclic T.I. Sylow $p$-subgroup $P$. Then $G$ is isomorphic to one of the following groups.

(a) $\text{PSL}_2(q)$ or $\text{PSU}_3(q)$, where $q = p^n$ and $n \geq 2$ or $n \geq 1$, respectively.
(b) $p = 2$ and $G \cong 2B_2(2^{2m+1})$.
(c) $p = 3$ and $G \cong 7G_2(3^{2m+1})$, where $m \geq 1$.
(d) $p = 3$ and $G \cong \text{PSL}_3(4)$ or $M_{11}$.
(e) $p = 5$ and $G \cong 2F_4(2)'$ or $M_c$.
(f) $p = 11$ and $G \cong J_4$.

**Proof.** If $p = 2$, then (a) and (b) follow from Theorem 1 of [13].

Let $p$ be odd. By Remark 2.1 $G$ can only be one of the simple $L(V)$ occurring in the list of Proposition 2.2. Since $\mathbb{A}_p \times \mathbb{A}_p$ is a subgroup of $\mathbb{A}_2p$, Lemma 1.2 asserts that $G \neq \mathbb{A}_2p$. A group $H$ with a T.I. Sylow $p$-subgroup has only $p$-blocks of defect zero and of highest defect. By the character table system CAS [11] $M(22)$ has a 5-block of defect one. Thus $G \neq M(22)$. Since $\text{Aut}(2B_2(2^5))$ is not simple, $G \neq \text{Aut}(2B_2(2^5))$. 


Now \( \text{PSL}_3(p^n) \) and \( \text{PSU}_3(p^n) \) have T.I. Sylow \( p \)-subgroups (see [5, pp. 191, 242]). By Ward [14] \( 2 \Gamma_2(3^{2m+1}) \) has a T.I. Sylow 3-subgroup. As can be seen from the character table of \( \text{PSL}_3(4) \) the Sylow 3-subgroup \( P \) equals \( C_G(x) \) for every \( 1 \neq x \in P \). Hence \( P \) is a T.I. set.

It is well known and easy to check that the Sylow 3-subgroups of \( M_{11} \) and the Sylow 5-subgroups of the Tits group \( 2 \Gamma_4(2)' \) and the McLaughlin group \( Mc \) are T.I. By Propositions 22 and 26 of Janko [7] the Sylow 11-subgroups of \( J_4 \) are T.I. This completes the proof.

3. Proof of the main result. In this section, Leonard’s conjecture is proved by means of the results mentioned above.

Let \( G \) be a finite group with a Sylow \( p \)-subgroup \( P \). If \( C_G(P) = C_G(x) \) for every \( 1 \neq x \in P \), then \( P \) is called weakly self-centralizing. The following lemma is well known.

**Lemma 3.1.** Let \( G \) be a finite group with a cyclic Sylow \( p \)-subgroup \( P \). Then \( P \) is a T.I. set if and only if \( P \) is weakly self-centralizing.

**Theorem 3.2.** Let \( G \) be a finite group with a T.I. Sylow \( p \)-subgroup \( P \). If \( G \) has a faithful complex character \( \chi \) with degree \( \chi(1) \leq \sqrt{|P|} - 1 \), then \( P \) is a normal subgroup of \( G \).

**Proof.** If \( P \) is cyclic, then \( P \) is weakly self-centralizing. As \( \sqrt{|P|} - 1 < \frac{1}{2}(|P| - 1) \) for every prime \( p > 0 \), it follows from Sibley’s theorem [12] that \( P < G \).

Now let \( G \) be a counterexample of minimal order. Then \( P \) is not cyclic, \( G = O^p(G) \), and by Proposition 1.4 \( Z = Z(G) = O_p(G) < O^p(G) = H \). Furthermore, \( H/Z \) is a nonabelian simple group with a T.I. Sylow \( p \)-subgroup, and we may assume that \( \chi \) is irreducible. We also can assume that \( G \) does not have a proper abelian direct factor.

Suppose that \( p \) is odd. Then \( m_p(G) > 1 \), because \( P \) is not cyclic. By Remark 2.1 \( N_G(P) \) is strongly \( p \)-embedded in \( G \). Therefore it follows from Propositions 2.2 and \( p = 3 \) and \( G \cong 2 \Gamma_2(3) \) or \( 2.3 \) that \( H = O^p(G) = G \) except when \( p = 3 \) and \( G \cong 2 \Gamma_2(3) \) or \( p = 5 \) and \( G/Z \cong \text{Aut}(2B_2(2^5)) \). Now remembering that \( p \nmid |Z| \) and using Gorenstein’s table [3, Table 4.1, p. 302] of the Schur multipliers of the finite simple groups, the structure of \( G \) can be described as in the following table.

<table>
<thead>
<tr>
<th>prime ( p )</th>
<th>( G/Z )</th>
<th>( Z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p \mid q )</td>
<td>PSL(2)(q) or PSU(3)(q)</td>
<td>(</td>
</tr>
<tr>
<td>( p = 3 )</td>
<td>( 2 \Gamma_2(3^{2m+1}) )</td>
<td>( Z = 1 )</td>
</tr>
<tr>
<td>( p = 3 )</td>
<td>PSL(3)(4) or ( M_{11} )</td>
<td>(</td>
</tr>
<tr>
<td>( p = 5 )</td>
<td>( 2 \Gamma_4(2)' )</td>
<td>( Z = 1 )</td>
</tr>
<tr>
<td>( p = 5 )</td>
<td>( \text{Aut}(2B_2(2^5)) )</td>
<td>( Z = 1 )</td>
</tr>
<tr>
<td>( p = 5 )</td>
<td>( Mc )</td>
<td>(</td>
</tr>
<tr>
<td>( p = 11 )</td>
<td>( J_4 )</td>
<td>( Z = 1 )</td>
</tr>
</tbody>
</table>
Applying the theorem of Landazuri and Seitz [8] on the minimal degrees of the nontrivial complex projective representations $\pi$ of $\text{PSL}_2(q)$, $\text{PSU}_3(q)$, or $^2G_2(3^{2m+1})$ we see that $\pi(1) \geq \frac{1}{2}(q - 1)$, $\pi(1) \geq q(q - 1)$, and $\pi(1) \geq 3^{2m+1}(3^{2m+1} - 1)$, respectively. In any case $\pi(1) > \sqrt{|P|} - 1$, a contradiction. If $p = 3$ and $G/Z \in \{\text{PSL}_3(4), M_{11}\}$, then $|P| = 9$. But another contradiction is obtained since the nontrivial irreducible projective characters of these simple groups have minimal degrees

$$\chi(1) = \begin{cases} 
4, & \text{if } G/Z = \text{PSL}_3(4), \\
10, & \text{if } G = M_{11}.
\end{cases}$$

If $p = 5$ and $G = ^2F_4(2)'$ then every nontrivial irreducible character $\chi$ of $G$ has degree $\chi(1) \geq 22$. However, $|P| = 25$, a contradiction.

If $p = 5$ and $G = \text{Aut}(^2B_2(2^3))$, then every faithful irreducible character $\chi$ of $G$ has degree $\chi(1) \geq \pi(1)$, where $\pi$ is a nontrivial irreducible character of the Suzuki group $^2B_2(q)$, $q = 2^3$, of minimal degree. Now by Landazuri and Seitz [8, p. 419], $\pi(1) = 4 \cdot 31 = 124$. Since $|P| = 125$, we obtain $\chi(1) > \sqrt{|P|} - 1$, a contradiction.

If $p = 5$ and $G/Z = M_{23}$, then every irreducible nontrivial character $\chi$ of $G$ has degree $\chi(1) \geq 22$ by the character table of $M_{23}$ (see [11]). As $|P| = 125$, $G$ cannot be a minimal counterexample. If $G/Z \equiv M_{23}$ and $|Z| = 3$, we again use the character table of $G$ (see [11]) and find that the nontrivial projective irreducible character $\chi$ of minimal degree has degree $\chi(1) = 126 \geq \sqrt{125} - 1$, another contradiction.

If $p = 11$ and $G = J_4$, then $\chi(1) \geq 1333$ by [11]. Since $|P| = 11^3$, $\chi(1) > \sqrt{|P|} - 1$, which is impossible by hypothesis.

Therefore $p = 2$. Hence by Theorem 2 of Suzuki [13] and Proposition 1.4 we get $G/Z \in \{\text{PSL}_2(q), \text{PSU}_3(q), ^2B_2(q)\}$, where $q$ is a power of 2. Using the theorem of Landazuri and Seitz [8] as above, we obtain our final contradiction. This completes the proof.

**Remark 3.3.** It is not possible to replace the bound $\sqrt{|P|} - 1$ by the bound $\frac{1}{2}(|P| - 1)$ of Sibley’s theorem [12]. Let $G = M_{23}$, $p = 5$, and $\chi$ be the irreducible character of $G$ with degree $\chi(1) = 22$. The Sylow 5-subgroup $P$ of $G$ is T.I. and has order $|P| = 5^3 = 125$. Hence $\chi(1) = 22 < 62 = \frac{1}{2}(|P| - 1)$. However, $P$ is not normal. In particular, Sibley’s condition that $P$ be weakly self-centralizing cannot be weakened to $P$ being a T.I. set.

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