A SHORT PROOF OF CAUCHY'S POLYGONAL NUMBER THEOREM

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Abstract. This paper presents a simple proof that every nonnegative integer is the sum of \( m + 2 \) polygonal numbers of order \( m + 2 \).

Let \( m \geq 1 \). The polygonal numbers of order \( m + 2 \) are the integers

\[
p_m(k) = \frac{m}{2} (k^2 - k) + k
\]

for \( k = 0, 1, 2, \ldots \). Fermat \([3]\) asserted that every nonnegative integer is the sum of \( m + 2 \) polygonal numbers of order \( m + 2 \). For \( m = 2 \), Lagrange \([5]\) proved that every nonnegative integer is the sum of four squares \( p_2(k) = k^2 \). For \( m = 1 \), Gauss \([4]\) proved that every nonnegative integer is the sum of three triangular numbers \( p_1(k) = (k^2 + k)/2 \), or, equivalently, that every positive integer \( n \equiv 3 \pmod{8} \) is the sum of three odd squares. Cauchy \([1]\) proved Fermat's statement for all \( m \geq 3 \), and Legendre \([6]\) refined and extended this result. For \( m \geq 3 \) and \( n \leq 120m \), Pepin \([8]\) published tables of explicit representations of \( n \) as a sum of \( m + 2 \) polygonal numbers of order \( m + 2 \), at most four of which are different from 0 or 1. Dickson \([2]\) prepared similar tables. Pall \([7]\) obtained important related results on sums of values of a quadratic polynomial.

Uspensky and Heaslet \([9, \text{p. 380}]\) and Weil \([10, \text{p. 102}]\) have written that there is no short and easy proof of Cauchy's polygonal number theorem. The object of this note is to present a short and easy proof.

Because of Pepin's and Dickson's tables, it suffices to consider only \( n \geq 120m \). For completeness, I also include a proof of Cauchy's lemma.

Cauchy's Lemma. Let \( a \) and \( b \) be odd positive integers such that \( b^2 < 4a \) and \( 3a < b^2 + 2b + 4 \). Then there exist nonnegative integers \( s, t, u, v \) such that

\[
\begin{align*}
(1) & \quad a = s^2 + t^2 + u^2 + v^2, \\
(2) & \quad b = s + t + u + v.
\end{align*}
\]

Proof. Since \( a \) and \( b \) are odd, it follows that \( 4a - b^2 \equiv 3 \pmod{8} \), and so, by Gauss's triangular number theorem, there exist odd integers \( x \geq y \geq z > 0 \) such that

\[
(3) \quad 4a - b^2 = x^2 + y^2 + z^2.
\]
Choose the sign of $\pm z$ so that $b + x + y \pm z \equiv 0 \pmod{4}$. Define integers $s, t, u, v$ by

$$s = \frac{b + x + y \pm z}{4}, \quad t = \frac{b + x}{2} - s = \frac{b + x - y \mp z}{4},$$

$$u = \frac{b + y}{2} - s = \frac{b - x + y \mp z}{4}, \quad v = \frac{b \mp z}{2} - s = \frac{b - x - y \pm z}{4}.$$ 

Then equations (1) and (2) are satisfied, and $s \geq t \geq u \geq v$. To show these integers are nonnegative, it suffices to prove that $v \geq 0$, or $v > -1$. This is true if $b - x - y - z > -4$, or, equivalently, if $x + y + z < b + 4$. The maximum value of $x + y + z$ subject to the constraint (3) is $\sqrt{12a - 3b^2}$, and the inequality $3a < b^2 + 2b + 4$ implies that $x + y + z \leq \sqrt{12a - 3b^2} < b + 4$. This proves the lemma.

**Theorem 1.** Let $m \geq 3$ and $n \geq 120m$. Then $n$ is the sum of $m + 1$ polygonal numbers of order $m + 2$, at most four of which are different from 0 or 1.

**Proof.** Let $b_1$ and $b_2$ be consecutive odd integers. The set of numbers of the form $b + r$, where $b \in \{b_1, b_2\}$ and $r \in \{0, 1, \ldots, m - 3\}$, contains a complete set of residue classes modulo $m$, and so $z = b + r \pmod{m}$ for some $b \in \{b_1, b_2\}$ and $r \in \{0, 1, \ldots, m - 3\}$. Define

$$a = 2\left(\frac{n - b - r}{m}\right) + b = \left(1 - \frac{2}{m}\right)b + 2\left(\frac{n - r}{m}\right).$$

Then $a$ is an odd integer, and

$$n = \frac{m}{2}(a - b) + b + r.$$

If $0 < b < \frac{3}{2} + \sqrt{6(n/m) - 3}$, then the quadratic formula implies that

$$b^2 - 4a = b^2 - 4\left(1 - \frac{2}{m}\right)b - 8\left(\frac{n - r}{m}\right) < 0$$

and so $b^2 < 4a$. Similarly, if $b > \frac{3}{2} + \sqrt{6(n/m) - 3}$, then $3a < b^2 + 2b + 4$. Since the length of the interval

$$I = \left[\frac{1}{2} + \sqrt{6\left(\frac{n}{m}\right) - 3}, \frac{2}{3} + \sqrt{6\left(\frac{n}{m}\right) - 8}\right]$$

is greater than 4, it follows that $I$ contains two consecutive odd positive integers $b_1$ and $b_2$. Thus, there exist odd positive integers $a$ and $b$ that satisfy (5) and the inequalities $b^2 < 4a$ and $3a < b^2 + 2b + 4$. Cauchy's Lemma implies that there exist $s, t, u, v$ satisfying (1) and (2), and so

$$n = \frac{m}{2}(a - b) + b + r = \frac{m}{2}(s^2 - s) + s + \cdots + \frac{m}{2}(v^2 - v) + v + r = p_m(s) + p_m(t) + p_m(u) + p_m(v) + r.$$ 

This completes the proof.

Note that this result is slightly stronger than Cauchy's theorem. Legendre [6] proved that every sufficiently large integer is the sum of five polygonal numbers of order $m + 2$, one of which is either 0 or 1. This can also be easily proved.
Theorem 2. Let $m \geq 3$. If $m$ is odd, then every sufficiently large integer is the sum of four polygonal numbers of order $m + 2$. If $m$ is even, then every sufficiently large integer is the sum of five polygonal numbers of order $m + 2$, one of which is either 0 or 1.

Proof. There is an absolute constant $c$ such that if $n > cm^3$, then the length of the interval $I$ defined in (6) is greater than $2m$, and so $I$ contains at least $m$ consecutive odd integers.

If $m$ is odd, these form a complete set of residues modulo $m$, and so $n \equiv b \pmod{m}$ for some odd number $b \in I$. Let $r = 0$. Define $a$ by formula (4).

If $m$ is even and $n > cm^3$, then $n \equiv b + r \pmod{m}$ for some odd integer $b \in I$ and $r \in \{0, 1\}$. Define $a$ by (4).

In both cases, the theorem follows immediately from Cauchy's Lemma.

References

2. L. E. Dickson, All positive integers are sums of values of a quadratic function of $x$, Bull. Amer. Math. Soc. 33 (1927), 713–720.
7. G. Pall, Large positive integers are sums of four or five values of a quadratic function, Amer. J. Math. 54 (1932), 66–78.

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