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AN UPPER BOUND FOR THE PERMANENT OF A 3-DIMENSIONAL $(0,1)$-MATRIX

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ABSTRACT. Let $A = (a_{ijk})$ be a 3-dimensional matrix of order n. The permanent of A is defined by

$$\operatorname{per} A = \sum_{\sigma,\tau \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)\tau(i)},$$

where S_n is the symmetric group on $\{1, 2, \ldots, n\}$. Suppose that A is a $(0,1)$-matrix and that $r_i = \sum_{j,k=1}^{n} a_{ijk}$ for $i = 1, 2, \ldots, n$. In this paper it is shown that $\operatorname{per} A \leq \prod_{i=1}^{n} r_i!^{1/r_i}$. A similar bound is then obtained for a second function, the 2-permanent of a 3-dimensional matrix, that is another analogue of the permanent of an ordinary (2-dimensional) matrix.

1. Introduction. It was conjectured by Minc [**3**] and proved by Brègman [**1**] that if A is a $(0,1)$-matrix of order n with row sums $r_1, r_2, \ldots, r_n$, then

$$\operatorname{per} A \leq \prod_{i=1}^{n} r_i!^{1/r_i},$$

where by definition $0!^{1/0} = 0$. In this paper it is shown that an analogous result holds for 3-dimensional matrices. The *permanent* of a 3-dimensional matrix $A = (a_{ijk})$ of order n is defined by

$$\operatorname{per} A = \sum_{\sigma,\tau \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)\tau(i)} \tag{1}$$

where S_n is the symmetric group on $\{1, 2, \ldots, n\}$. We now state our principal result.

THEOREM 1. *Let $A = (a_{ijk})$ be a 3-dimensional $(0,1)$-matrix of order n, and let $r_i = \sum_{j,k=1}^{n} a_{ijk}$ for $i = 1, 2, \ldots, n$. Then*

$$\operatorname{per} A \leq \prod_{i=1}^{n} r_i!^{1/r_i}.$$

The proof of Theorem 1, presented in §2, is an adaptation of Schrijver's proof [**4**] of the Minc-Brègman bound, with the extension to 3-dimensional matrices requiring a number of additional lemmas which may be of some independent interest.

In equation (1) and Theorem 1 the planes of A play the role of the rows and columns of an ordinary (2-dimensional) matrix, where a *plane* of a 3-dimensional

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matrix $A = (a_{ijk})$ is a submatrix obtained by fixing one of the indices i, j, k and letting the other two vary. We could also have taken the analogue of rows and columns to be the *lines* of A, obtained by fixing two of these indices. In §3 we give a second definition of the permanent of a 3-dimensional matrix, called the 2-permanent, based on this observation. Using a result from [**2**] which relates these two types of permanents we obtain an upper bound for the 2-permanent of a 3-dimensional $(0,1)$-matrix similar to the one given in Theorem 1, with plane sums replaced by line sums.

2. Proof of the principal result. We write N, Z, and R^+ to denote, respectively, the set of natural numbers, the set of integers, and the set of positive real numbers. Also, for any real number x, we use the notation $\lfloor x \rfloor = \max\{k \in Z : k \leq x\}$ and $\lceil x \rceil = \min\{k \in Z : x \leq k\}$. The first lemma appears in [**4**].

LEMMA 1. *If $t = \sum_{i=1}^{n} t_i$, $t_i \in R^+$, then $t^t \leq n^t \prod_{i=1}^{n} t_i^{t_i}$.*

LEMMA 2. *For all $n \in N$, $n!^{1/n}(n+2)!^{1/(n+2)} \leq (n+1)!^{2/(n+1)}$.*

PROOF. It is well known that if $x_1, x_2, \ldots, x_k \in R^+$, then $\prod_{i=1}^{k} x_i \leq \overline{x}^k$ where $\overline{x} = (\sum_{i=1}^{k} x_i)/k$. Let $x_i = i$ for $i = 1, 2, \ldots, n$, $x_i = i - n$ for $i = n+1, n+2, \ldots, 2n$, and $x_i = n+2$ for $i = 2n+1, 2n+2, \ldots, n^2+3n$. Then

$$\prod_{i=1}^{n^2+3n} x_i = n!^2 (n+2)^{n^2+n}$$

and

$$\sum_{i=1}^{n^2+3n} x_i = (n^2+3n)(n+1).$$

Hence the inequality above becomes $n!^2(n+2)^{n^2+n} \leq (n+1)^{n^2+3n}$, which is equivalent to the desired inequality.

LEMMA 3. *Suppose that a function $f\colon N \to R^+$ has the property that*

$$f(x)f(x+2) \leq (f(x+1))^2$$

for all $x \in N$, and that $x_1, x_2, \ldots, x_k \in N$. Let $\sum_{i=1}^{k} x_i = k\overline{x} = k_1 \lfloor \overline{x} \rfloor + k_2 \lceil \overline{x} \rceil$, where $k_1, k_2 \in Z$, $k_1, k_2 \geq 0$, and $k_1 + k_2 = k$. Then

$$\prod_{i=1}^{k} f(x_i) \leq f(\lfloor \overline{x} \rfloor)^{k_1} f(\lceil \overline{x} \rceil)^{k_2}.$$

PROOF. It follows from induction on n that

$$f(x)f(x+n) \leq f(x+1)f(x+n-1) \tag{2}$$

for all x, $n \in N$. To see this note that the inductive step is

$$f(x)f(x+n+1) = \frac{f(x)f(x+2)f(x+1)f(x+1+n)}{f(x+1)f(x+2)} \leq f(x+1)f(x+n).$$

By (2), if there exist p and q such that $x_p - x_q > 1$ and we let $x_p' = x_p - 1$, $x_q' = x_q + 1$, and $x_i' = x_i$ for $i \neq p, q$, then $\prod_{i=1}^{k} f(x_i) \leq \prod_{i=1}^{k} f(x_i')$. By repeating this step the upper bound eventually becomes $f(\lfloor \overline{x} \rfloor)^{k_1} f(\lceil \overline{x} \rceil)^{k_2}$.

LEMMA 4. *Let $f(x) = x!^{1/x}$ for $x \in N$. Suppose that $n, p, r \in N$, $n \geq 3$, and $p \leq (r-1)/(n-1)$. Then*

$$f(r-p-1)^{r-pn+p-1} f(r-p)^{(p+1)n-p-r} \leq (r-1)!r!^{n/r-1}.$$

PROOF. Let $k \in N$, $k \leq p$. Then $r-k-1 > 0$. Therefore, by Lemma 2, $f(r-k-1)f(r-k+1) \leq f(r-k)^2$. Since $k \leq (r-1)/(n-1)$, we can raise both sides of this inequality to the power $r-kn+k-1$ to obtain

$$f(r-k-1)^{r-kn+k-1} f(r-k+1)^{r-kn+k-1} \leq f(r-k)^{2(r-kn+k-1)}.$$

This inequality is equivalent to

$$\begin{aligned} (3) \qquad & f(r-k-1)^{r-kn+k-1} f(r-k)^{(k+1)n-k-r} \\ & \leq f(r-k)^{r-(k-1)n+k-2} f(r-k+1)^{kn-k+1-r}. \end{aligned}$$

The desired inequality can now be obtained by successively applying (3) with $k = p$, $p-1$, $p-2, \ldots, 1$.

LEMMA 5. *Let $x_1, x_2, \ldots, x_{n-1}$, $r \in N$ with $n > 2$. If $\sum_{i=1}^{n-1} x_i \leq (n-2)r+1$, then*

$$\prod_{i=1}^{n-1} x_i!^{1/x_i} \leq (r-1)!r!^{n/r-1}.$$

PROOF. Since $x!^{1/x}$ is increasing in x we may assume that $\sum_{i=1}^{n-1} x_i = (n-2)r+1$. Hence

$$\overline{x} = \frac{(n-2)r+1}{n-1} = r - \frac{r-1}{n-1}.$$

Let $p = \lfloor (r-1)/(n-1) \rfloor$. Then $\lfloor \overline{x} \rfloor = \lceil \overline{x} \rceil = r-p$ if x is an integer and $\lfloor \overline{x} \rfloor = r-p-1$, $\lceil \overline{x} \rceil = r-p$ otherwise. Since

$$(r-pn+p-1)(r-p-1) + ((p+1)n-p-r)(r-p) = (n-2)r+1$$

we can apply Lemma 3 with $f(x) = x!^{1/x}$, $k = n-1$, $k_1 = r-pn+p-1$, and $k_2 = (p+1)n-p-r$. (When $\lfloor \overline{x} \rfloor = \lceil \overline{x} \rceil = r-p$ we have $k_1 = 0$.) Therefore Lemmas 3 and 4 imply that

$$\prod_{i=1}^{n-1} x_i!^{1/x_i} \leq f(r-p-1)^{r-pn+p-1} f(r-p)^{(p+1)n-p-r} \leq (r-1)!r!^{n/r-1}.$$

LEMMA 6. *If $B = (b_{jk})$ is a $(0,1)$-matrix of order n with $\sum_{j,k=1}^{n} b_{jk} = r$, then*

$$\sum_{i=1}^{n-1} \sum_{\substack{j,k \\ j \neq i \\ k \neq i}} b_{jk} \leq (n-2)r+1.$$

PROOF.

$$\begin{aligned} \sum_{i=1}^{n-1} \sum_{\substack{j,k \\ j \neq i \\ k \neq i}} b_{jk} &= \sum_{i=1}^{n-1} \left(r - \sum_{j=1}^{n} b_{ji} - \sum_{k=1}^{n} b_{ik} + b_{ii} \right) \\ &= (n-3)r + \sum_{j=1}^{n} b_{jn} + \sum_{k=1}^{n} b_{nk} + \sum_{i=1}^{n-1} b_{ii} \\ &\leq (n-3)r + r + 1 = (n-2)r+1. \end{aligned}$$

We now show how Schrijver's proof [4] of the Minc-Brègman bound can be altered to prove Theorem 1.

PROOF OF THEOREM 1. The proof is by induction on n. Clearly the theorem holds for $n = 1$. Let $n > 1$. It suffices to assume that $r_i \in N$ for $i = 1, 2, \ldots, n$. Let A_{ijk} denote the 3-dimensional submatrix of A of order $n-1$ obtained by deleting the three planes through cell (i, j, k). Let

$$S = \left\{ (\sigma, \tau) : \prod_{i=1}^{n} a_{i\sigma(i)\tau(i)} = 1 \right\}.$$

Then $|S| = \operatorname{per} A$. Apply Lemma 1 with $t = \operatorname{per} A = \sum_{j,k;a_{ijk}=1} \operatorname{per} A_{ijk}$ for $i = 1, 2, \ldots, n$, to obtain

$$\begin{aligned}(\operatorname{per} A)^{n \operatorname{per} A} &= \prod_{i=1}^{n} (\operatorname{per} A)^{\operatorname{per} A} \\ &\leq \prod_{i=1}^{n} r_i^{\operatorname{per} A} \prod_{\substack{j,k \\ a_{ijk}=1}} (\operatorname{per} A_{ijk})^{\operatorname{per} A_{ijk}}.\end{aligned}$$

For each i we have

$$\begin{aligned}\prod_{\substack{j,k \\ a_{ijk}=1}} (\operatorname{per} A_{ijk})^{\operatorname{per} A_{ijk}} &= \prod_{\substack{j,k \\ a_{ijk}=1}} (\operatorname{per} A_{ijk})^{|\{(\sigma,\tau)\in S : \sigma(i)=j, \tau(i)=k\}|} \\ &= \prod_{(\sigma,\tau)\in S} \operatorname{per} A_{i\sigma(i)\tau(i)}.\end{aligned}$$

By the induction hypothesis,

$$\operatorname{per} A_{i\sigma(i)\tau(i)} \leq \prod_{\substack{p \\ p\neq i}} r(p,i)!^{1/r(p,i)}$$

where

$$r(p,i) = \sum_{\substack{j,k \\ j\neq\sigma(i) \\ k\neq\tau(i)}} a_{pjk}.$$

Combining the inequalities above we have

$$\begin{aligned}(\operatorname{per} A)^{n \operatorname{per} A} &\leq \prod_{i=1}^{n} r_i^{\operatorname{per} A} \prod_{(\sigma,\tau)\in S} \prod_{\substack{p \\ p\neq i}} r(p,i)!^{1/r(p,i)} \\ &= \prod_{(\sigma,\tau)\in S} \left(\prod_{i=1}^{n} r_i \right) \prod_{i=1}^{n} \prod_{\substack{p \\ p\neq i}} r(p,i)!^{1/r(p,i)} \\ &= \prod_{(\sigma,\tau)\in S} \left(\prod_{i=1}^{n} r_i \right) \prod_{p=1}^{n} \prod_{\substack{i \\ i\neq p}} r(p,i)!^{1/r(p,i)}.\end{aligned}$$

Fix σ, τ, and p and let $b_{jk} = a_{p\sigma(j)\tau(k)}$. Then $\sum_{j,k} b_{jk} = r_p$ and for each i,

$$\sum_{\substack{j,k \\ j\neq i, k\neq i}} = r(p,i).$$

Hence, by Lemma 6,

$$\sum_{\substack{i \\ i\neq p}} r(p,i) \leq (n-2)r_p + 1.$$

Therefore, by Lemma 5, if $n > 2$, then

$$\prod_{\substack{i \\ i\neq p}} r(p,i)!^{1/r(p,i)} \leq (r_p - 1)! r_p!^{n/r_p - 1}.$$

Moreover, it is easy to see that this inequality also holds for $n = 2$. Therefore,

$$\begin{aligned}(\operatorname{per} A)^{n \operatorname{per} A} &\leq \prod_{(\sigma,\tau)\in S} \left(\prod_{i=1}^{n} r_i\right) \prod_{p=1}^{n} (r_p - 1)! r_p!^{n/r_p - 1} \\ &= \prod_{(\sigma,\tau)\in S} \prod_{p=1}^{n} r_p!^{n/r_p} = \left(\prod_{i=1}^{n} r_i!^{1/r_i}\right)^{n \operatorname{per} A},\end{aligned}$$

and the theorem follows.

For some values of the r_i it is easy to construct matrices A such that equality holds in Theorem 1. Let $A_m = (a_{ijk})$ denote the 3-dimensional $(0,1)$-matrix of order m with $a_{ijj} = 1$ for $i, j = 1, 2, \ldots, m$, and $a_{ijk} = 0$ otherwise. Then $r_i = \sum_{j,k=1}^{n} a_{ijk} = m$ for $i = 1, 2, \ldots, m$, and $\operatorname{per} A_m = m!$. Now let $n_1, n_2, \ldots, n_t$ be a partition of n, and let A be the direct sum of $A_{n_1}, A_{n_2}, \ldots, A_{n_t}$. It follows that A is a 3-dimensional $(0,1)$-matrix of order n such that equality holds in Theorem 1.

3. The 2-permanent. Theorem 1 can be used to obtain another extension of the Minc-Brègman inequality to 3-dimensional matrices. Let $A = (a_{ijk})$ be a 3-dimensional matrix of order n. The planes of A are the submatrices obtained by fixing one of i, j, k, and the lines of A are the submatrices obtained by fixing two of i, j, k. Observe that the permanent (or 1-*permanent*) of A is equal to the summation of all products of n entries of A, no two entries from the same plane. Similarly, the 2-*permanent* of A is defined to be the summation of all products of n^2 entries of A, no two entries from the same line [**2**]. We have the following upper bound for the 2-permanent of a $(0,1)$-matrix.

THEOREM 2. *Let $A = (a_{ijk})$ be a 3-dimensional $(0,1)$-matrix of order n, and let $r_{ij} = \sum_{k=1}^{n} a_{ijk}$ for $i, j = 1, 2, \ldots, n$. Then*

$$2\text{-}\operatorname{per} A \leq \prod_{i,j=1}^{n} r_{ij}!^{1/r_{ij}}.$$

PROOF. As shown by the authors [**2**], there exists a 3-dimensional $(0,1)$-matrix $B = (b_{ijk})$ of order n^2 such that $\operatorname{per} B = 2\text{-}\operatorname{per} A$. Moreover, B can be chosen so that if $s_i = \sum_{j,k=1}^{n^2} b_{ijk}$ for $i = 1, 2, \ldots, n^2$, then $\{s_i : i = 1, 2, \ldots, n^2\} = \{r_{ij} : i, j = 1, 2, \ldots, n\}$. Therefore, if Theorem 1 is applied to B the desired upper bound on $2\text{-}\operatorname{per} A$ is obtained.

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