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AUTOMATIC CONTINUITY AND OPENNESS OF CONVEX RELATIONS

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ABSTRACT. We establish open mapping and lower semicontinuity results for convex relations which are generalizations of classical results on continuity of positive linear mappings. These results do not require closure of the relation. Several applications are given.

1. Preliminaries. Let X and Y be (separated, real) topological vector spaces. A relation $H\colon X \to Y$ is *convex* if its *graph* $\operatorname{Gr}(H) := \{(x,y)|y \in H(x)\}$ is convex in $X \times Y$. The relation is a *convex process* if $\operatorname{Gr}(H)$ is actually a convex cone. We denote the *domain* by $D(H) := \{x|H(x) \neq \emptyset\}$ and the *range* by $R(H) := \{y|y \in H(x), \exists x \in X\}$. The *inverse* relation is defined by $x \in H^{-1}(y)$ if $y \in H(x)$, and has $R(H^{-1}) = D(H)$. For any set C in X, $H(C) := \{H(x)|x \in C\}$ so that $R(H) = H(X)$.

A relation H is *LSC at x* in $D(H)$ if for each y in $H(x)$, $x \in \operatorname{int} H^{-1}(N+y)$ for any neighborhood N of zero in Y. A relation H is *open at y* in $R(H)$ if for each x in $H^{-1}(y)$, $y \in \operatorname{int} H(N+x)$ for any neighborhood N of zero in X. Then clearly H is open at y if and only if H^{-1} is LSC at y. These matters are described in detail in [**1**, **2**, **3**, **7**, **9**].

Now suppose that S is a convex cone in Y. We say $f\colon X \to Y$ is *S-convex* exactly when the relation $H_f(x) := f(x) + S$ is convex. Then, allowing $f(x)$ to be plus infinity, we have the *domain* of f, $\operatorname{dom}(f) := D(H_f)$. We recall that S is *generating* if $S - S = Y$ and *normal* if there is a neighborhood base at 0 consisting of sets V such that $(S - V) \cap (V - S) = V$. When S is normal, H_f is LSC at x if and only if f is continuous at x [**2**]. If $S \subset X$ and $K \subset Y$ are convex cones, then we say f is *isotone with respect* to S and K if $f(x_1) - f(x_2) \in K$ whenever $x_1 - x_2 \in S$ and x_1, x_2 are in $\operatorname{dom}(f)$. We recall that the *core* of a set A, $\operatorname{core}(A)$, is the set of points $\bar{a}$ in A such that for all x in X, $x \in \lambda(A - \bar{a})$ for λ sufficiently large (depending on x). If $0 \in \operatorname{core}(A)$ we say A is *absorbing*.

We denote the continuous *dual cone* of S in X by $S^+ := \{x' \in X'|x'(x) \geq 0 \text{ for } x \text{ in } S\}$. Here X' is the continuous dual of X.

2. The main result.

DEFINITION 2.1. Let X and Y be linear spaces.

(a) Let $S \subset X$ be a convex cone. We say $H\colon X \to Y$ is an *S-isotone relation* if for x in X

$$H(x+S) \subset H(x), \tag{2.1}$$

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or equivalently, if $H(x_1) \subset H(x_2)$ whenever $x_1 \geq_S x_2$ $(x_1 - x_2 \in S)$.

(b) Let $S \subset Y$ be a convex cone. We say that $H: X \to Y$ is an *S-invariant relation* if for x in X

$$H(x) - S \subset H(x). \tag{2.2}$$

Then H is S-isotone if and only if H^{-1} is S-invariant (reversing the roles of X and Y).

THEOREM 2.2. *Let X and Y be topological vector spaces. Let $H: X \to Y$ be a convex relation.*

(a) *Let Y be complete and metrizable. Suppose that H is S-invariant for a closed generating cone S in Y. Then H is open throughout the core of $R(H)$.*

(b) *Let X be complete and metrizable. Suppose that H is S-isotone for a closed generating cone S in X. Then H is LSC throughout the core of $D(H)$.*

PROOF. From the previous discussion, we see that (a) and (b) are entirely dual (on replacing H by H^{-1}). Thus we need prove only (a). Let y_0 lie in $\operatorname{core}(R(H))$. Let $x_0 \in H^{-1}(y_0)$. On replacing H by $H(\cdot + x_0) - y_0$ [which leaves hypotheses and conclusions unchanged] we may suppose $x_0 = 0$ and $y_0 = 0$. Let U be any absorbing subset of X. Then $H(U)$ is "almost absorbing" in Y. Indeed, since H is convex and $0 \in H(0)$, for u in X we have

$$2^{-n}H(2^n u) \subset 2^{-n}H(2^n u) + (1 - 2^{-n})H(0) \subset H(u), \qquad n \in \mathbf{N}.$$

Thus

$$R(H) = \bigcup_{\mathbf{N}} H(2^n U) \subset \bigcup_{\mathbf{N}} 2^n H(U),$$

since U is absorbing. As $R(H)$ is absorbing,

$$Y = \bigcup_{n=1}^{\infty} 2^n H(U). \tag{2.3}$$

Let $(W_n)_{\mathbf{N}}$ be a neighborhood base at zero in Y such that $W_{n+1} + W_{n+1} \subset W_n$ for n in $\mathbf{N}$. Let

$$V_n := 2^{-n}[W_n \cap S - S]. \tag{2.4}$$

Then V_n is a zero neighborhood. Indeed, the convex relation $Q: Y \to Y$ defined by $Q(y) := y - S$ if $y \in S$, is closed with $R(Q) = Y$. Since Y is complete metrizable, the Fréchet open mapping theorem [**1**, **4**] shows that $0 \in \operatorname{int} Q(W_n)$ for each n. This also follows from Klee's result [**5**, p. 194]. If some V_n lies in $H(U)$ we are done.

If not, select v_n in V_n with $v_n \notin H(U)$ for each n. Then (2.4) shows that $v_n \leq_S 2^{-n} w_n$ for some w_n in $W_n \cap S$. Let $z_m := \sum_{n=0}^{m} w_n$.

Then inductively, $z_{m+p} - z_m \in W_m \cap S$ for m and p in $\mathbf{N}$. Hence $(z_n)_{\mathbf{N}}$ is Cauchy and converges to some z in S, as S is closed. Now $2^{-n}z \geq_S 2^{-n}w_n \geq_S v_n$ and so $v_n \in 2^{-n}z - S$. Then (2.2) shows that $2^{-n}z \notin H(U)$. This contradicts (2.3). Thus H is open at y_0. $\square$

REMARK 2.3. We have in fact established slightly more in (a): if $y_0 \in \operatorname{core}(R(H))$, then

$$y_0 \in \operatorname{int} H(x_0 + U) \tag{2.5}$$

for any x_0 in $H^{-1}(y_0)$ and any absorbing set U in X (and similarly in (b)).

COROLLARY 2.4. *Let X and Y be topological vector spaces. Suppose that X is complete metrizable and $S \subset X$ is a closed generating convex cone. Suppose that $f\colon X \to Y \cup \{\infty\}$ is K-convex for some normal convex cone K in Y. Suppose also that f is isotone with respect to S and K. Then f is continuous throughout* core(dom(f)).

PROOF. $H_f(x) := f(x) + K$ is convex and S-isotone. Thus Theorem 2.2(b) shows that H_f is LSC on core($D(H)$). Since K is normal, f is actually continuous on core(dom(f)) [**2**, p. 191]. □

We futher specialize to recover the Nachbin-Namioka result given by Peressini [**5**, p. 86].

COROLLARY 2.5. *With X, Y, S, and K as above, suppose that $P\colon X \to Y$ is linear and suppose that P is positive: $P(S) \subset K$. Then P is continuous.*

PROOF. P is K-convex, isotone with respect to S and K, and dom$(P) = X$. □

A similar result to Theorem 2.2 is easily obtained when $\operatorname{int} S$ is nonempty.

PROPOSITION 2.6. *Suppose in Theorem* 2.2 *that* $\operatorname{int} S$ *is nonempty, but not necessarily closed, while X and Y are arbitrary topological vector spaces. Then the conclusions of Theorem* 2.2 *hold.*

PROOF. Again we show only (a). But now if $y_0 \in H(x_0)$,

$$y_0 - \operatorname{int} S \subset H(x_0) - S \subset H(x_0).$$

It follows that $\operatorname{int} H(x_0)$ is nonempty and that H is (strongly) open at some point. Then H is open throughout $\operatorname{int} R(H)$ [**2**, p. 191]. □

Similarly, all the corollaries still obtain.

COROLLARY 2.7. *Let X and Y be topological vector spaces with Y complete metrizable. Let $C \subset X$ be a convex set. Let $f\colon X \to Y \cup \{\infty\}$ be S-convex with respect to a closed generating cone S.*

(a) *Suppose that*

$$0 \in \operatorname{core}[f(C) + S]. \tag{2.6}$$

Then the convex relation H defined by

$$H(x) := \begin{cases} f(x) + S, & x \in C, \\ \varnothing, & \textit{else}, \end{cases} \tag{2.7}$$

is open at zero.

(b) *Let $p\colon X \to \mathbf{R} \cup \{\infty\}$ be convex with $H^{-1}(0) \cap \operatorname{core}(\operatorname{dom}(p)) \neq \varnothing$. Consider the convex program*

$$(\mathrm{CP}) \quad -\infty < \mu := \inf\{p(x) | f(x) \leq_S 0,\ x \in C\}. \tag{2.8}$$

Then Lagrange multipliers exist for (CP): *there is a continuous linear functional λ on Y such that*

$$\mu = \inf\{p(x) + \lambda f(x) | x \in C\}, \tag{2.9}$$

and $\lambda(S) \geq 0$.

PROOF. H is convex and $(-S)$-invariant. This establishes (a) as a consequence of Theorem 2.2(a).

(b) Let $0 \in H(x_0)$ with x_0 in core(dom(p)). Then $U := \{x | p(x+x_0) \le p(x_0)+1\}$ is absorbing in X. Then (2.5) shows that $0 \in \operatorname{int} H(x_0 + U) =: W$.

Let $h(w) := \inf\{p(x) | w \in H(x)\}$. Then, as always, h is convex and also $h(w) \le p(x_0) + 1$ for w in W. Since $h(0)$ is finite, h is continuous on U and as in [**1**, **8**] this suffices to establish (2.9). □

Note that there are no continuity or closure hypotheses on v, f, or C, other than those implicit in the closedness of S.

COROLLARY 2.8. *Let X and Y be topological vector spaces with Y complete metrizable. Let $K \subset X$ and $S \subset Y$ be convex cones with S closed and generating. Let $A\colon X \to Y$ be linear (possibly not even densely defined).*

(a) *The following are equivalent:*

$$\text{(2.10)} \qquad (1) \quad A(K) - S = Y,$$

$$\text{(2.11)} \qquad (2) \quad H_A(x) := \begin{cases} Ax - S, & x \in K, \\ \varnothing, & \textit{else}, \end{cases}$$

is an open convex process.

(b) *Both imply the Krein-Rutman identity* [**18**]:

$$\text{(2.12)} \qquad [K \cap A^{-1}(S)]^+ = K^+ + S^+A.$$

PROOF. Clearly, H_A is S-invariant and (a) follows.

(b) Suppose $\Phi \in [K \cap A^{-1}(S)]^+$. Then

$$0 = \inf\{\Phi(x) | -Ax \le_S 0,\ x \in K\}.$$

The previous corollary applies and we can write

$$\text{(2.13)} \qquad 0 = \inf\{\Phi(x) - \lambda(Ax) | x \in K\}$$

for some λ in S^+. But (2.13) shows $\Phi - \lambda A \in K^+$ and so $\Phi \in K^+ + S^+A$. The opposite inclusion always holds. □

Note that we can phrase a similar result to (2.12) without assuming Φ is continuous.

COROLLARY 2.9. *Let X and Y be as in Corollary* 2.8. *Let $H\colon X \to Y$ be a convex process.*

(a) *Suppose that $H(0)$ contains a closed, generating subcone S. Then H is open if and only if $R(H) = Y$.*

(b) *Suppose $H(0)$ is closed and $H(0) - H(0) = Y$. Then H is open if and only if $R(H) = Y$.*

PROOF. (a) We have

$$H(x) + S \subset H(x) + H(0) \subset H(x)$$

and H is $(-S)$-invariant.

(b) In this case H is $(-H(0))$-invariant. □

Of course, Corollary 2.8(a) is a special case of Corollary 2.9(b). There is an obvious dual version.

EXAMPLE 2.10. (a) Let X be the normed lattice, Φ, consisting of all finite sequences in the supremum norm. Let $S := K := \{x \in X | x_n \ge 0,\ n \in \mathbf{N}\}$ be the lattice cone. Let $P\colon X \to X$ be defined by $(Px)_n := nx_n$. Then P is positive linear and discontinuous. All hypotheses of Corollary 2.4 hold other than completeness.

(b) Closed generating cones include dual cones of normal cones, closed cones with nonempty interior, and all closed lattice orderings. Thus our central hypothesis is generally satisfied. □

COROLLARY 2.11. *Let Y be a complete metrizable topological vector space. Let $S \subset Y$ be a closed generating cone and let $V \subset Y$ be any absorbing set. Then the full hull of V,*

$$[V]_S := (V - S) \cap (S - V),$$

is a neighborhood of zero.

PROOF. $H(x) := x - S$ is convex and S-invariant on Y. Then Remark 2.3 shows that $0 \in \operatorname{int} H(V)$ and, by symmetry, $0 \in \operatorname{int}[H(V) \cap -H(V)]$. □

With Φ, S as in Example 2.10(a) we let $V := \{x \in \Phi \mid |x_n| \le 2^{-n},\ n \in \mathbf{N}\}$. Then V is absorbing and $[V]_S = V$ has no interior.

There is also a local version of Theorem 2.2, which may be useful in applications. We say that H is *locally S-isotone around* x_0 in $D(H)$ if there is a neighborhood N of x_0 in X such that $H(x_1) \subset H(x_2)$ when $x_2 \le_S x_1$ and $x_1, x_2 \in \mathbf{N}$. Similarly, H is *locally S-invariant around* y_0 in $R(H)$ if there is a neighborhood N of y_0 in Y such that if $y_1 \in H(x)$ and $y_2 \le_S y_1$ with y_1, y_2 in N, then $y_2 \in H(x)$. An analysis of the proof of Theorem 2.2 shows that (a) will prove that if H is locally S-invariant around y_0 in core $R(H)$, then H is open around y_0 (and similarly in (b)). It follows as in [**1**] that H is actually open throughout core$(R(H))$.

In a lattice setting it is natural to consider the following variant of isotonicity. We say H is *S-absolutely isotone* if

$$H(x_1) \subset H(x_2) \quad \textit{whenever} \quad |x_2| \le_S |x_1|.$$

This is equivalent to demanding that $H^{-1}(y)$ is *solid* for each y: if $x_1 \in H^{-1}(y)$ and $|x_2| \le_S |x_1|$, then $x_2 \in H^{-1}(y)$.

It is also easy to verify that H is absolutely isotone exactly when

$$H(x) = H(|x|), \textit{and } H(x_1) \subset H(x_2) \textit{ if } 0 \le_S x_2 \le_S x_1.$$

Moreover, if H is S-isotone and convex, then $H(|\cdot|)$ is S-absolutely isotone and convex.

Theorem 2.2 has an obvious analogue in this context, in which we assume that $|\cdot|$ is continuous.

3. A bornological variant. Recall that a locally convex space is *bornological* if every symmetric convex subset which absorbs bounded sets is a zero neighborhood. Every metrizable space is bornological, as are inductive limits and quotients of bornological spaces [**6**]. A convex cone S in X is a *strict b-cone* if each bounded set B in X lies inside a set of the form $S \cap B' - S \cap B'$ for some other bounded set B'. Normal and strict *b*-cones are effectively dual to each other [**5**]. We will say that S is *increasingly sequentially complete if* every S-isotone Cauchy sequence converges. This holds in any complete metrizable space, or in any semireflexive space.

THEOREM 3.1. *Let X and Y be locally convex spaces. Let $H\colon X \to Y$ be a convex relation.*

(a) *Let Y be bornological. Suppose that H is S-invariant with respect to an increasingly sequentially complete strict b-cone S in Y. Then H is open throughout* $\operatorname{core}(R(H))$.

(b) *Let X be bornological. Suppose that H is S-isotone with respect to an increasingly sequentially complete strict b-cone S in X. Then H is continuous throughout* $\operatorname{core}(D(H))$.

PROOF. Again (a) and (b) are equivalent and we will establish (a). As before we may assume that $y_0 = 0, x_0 = 0$, and $0 \in \operatorname{core}(R(H))$. Then let U be any convex absorbing subset of X. Again, it suffices to show that $0 \in \operatorname{int} H(U)$. Now $H(U)$ is convex because H and U are. As Y is real and bornological, it suffices to show that $H(U)$ absorbs bounded sets. Suppose not. Then we can find B bounded and $(b_n)_{\mathbf{N}} \subset B$ such that

$$4^{-n} b_n \notin H(U) \quad \text{for } n \in \mathbf{N}.$$

Since S is a strict b-cone, $B \subset S \cap B_0 - S$ for some bounded convex symmetric B_0. Let $(d_n)_{\mathbf{N}}$ lie in $S \cap B_0$ with $b_n \leq_S d_n$ for each n.

Let $z_n := \sum_{m=0}^{n} 2^{-m} d_m$. Then

$$z_{n+p} - z_n \in (2^{-n} B_0) \cap S$$

for n, p in $\mathbf{N}$. Thus $(z_n)_{\mathbf{N}}$ is Cauchy because B_0 is bounded, and increases. By assumption, (z_n) converges to z in S. Then

$$2^{-n} z \geq_S 4^{-n} d_n \geq_S 4^{-n} b_n \quad \text{for } n \in \mathbf{N}.$$

As H is S-invariant, $2^{-n} z \notin H(U)$ for each n. This contradicts (2.3). Thus $H(U)$ does absorb bounded sets. □

REMARK. (a) As before we have established more in (a): If $y_0 \in \operatorname{core}(R(H))$, then

$$y_0 \in \operatorname{int} H(x_0 + U) \tag{3.1}$$

for any x_0 in $H^{-1}(y_0)$ and any convex, absorbing subset U.

(b) Thus all the corollaries of §2 remain valid under the appropriate hypotheses of Theorem 3.1. In particular Corollary 2.5, so adjusted, recaptures Schaefer's result for linear operators [**5**, p. 86]. In Corollary 2.11, V must be supposed convex.

(c) As we have illustrated these open mapping results are applicable in a variety of situations. Thus, they should be a useful addition to the more standard closed graph type of results due to Robinson [**7**], Ursescu [**9**], Jameson [**4**], the author [**1**, **2**, **3**], and elsewhere.

This is particularly so in the context of Corollaries 2.7, 2.8, and 2.9 because we make no hypotheses on the domain, only on the perturbation space. Moreover, the range space conditions are very often met. □

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