ABSTRACT. It is proved in this note that the so-called A-property is necessary in order that the finite-dimensional space $U$ be Chebyshev in $C(K)$ with respect to the norm $\|f\| = \int_K \omega |f|$ for every positive continuous weight $\omega$. It is also shown that for each finite-dimensional subspace $U$ there exists a positive continuous weight $\omega$ such that $U$ is Chebyshev in $C(K)$ with respect to this weight $\omega$.

NOTATION. Let $K$ be a compact subset of $\mathbb{R}^n$ and denote by $C(K)$ the set of continuous real functions on $K$. Furthermore, we denote $W_C = \{\omega \in C(K): \omega > 0 \text{ on } K\}$ and $W = \{\omega: \omega \text{ is measurable and } 0 < \inf\{\omega(x): x \in K\} \leq \sup\{\omega(x): x \in K\} < \infty\}$. For any $\omega \in W$ we denote by $C_\omega(K)$ the space $C(K)$ endowed with the norm

\[ \|f\|_{L_\omega(K)} = \int_K \omega |f|. \]

Furthermore, we shall say that a finite-dimensional subspace $U$ in $C(K)$ is a Chebyshev subspace of $C_\omega(K)$ if every $f \in C(K)$ has a unique best approximant in $U$ with respect to norm (1). We shall assume that $\text{Int} K = K$.

By a result of Strauss [3] if $U$ satisfies the so-called A-property then $U$ is a Chebyshev subspace of $C_\omega(K)$ for every $\omega \in W$. In [1] we proved the converse of this statement: If $U$ is Chebyshev in $C_\omega(K)$ for every $\omega \in W$ then $U$ is an A-space. Independently, Pinkus [2] gave another version of this result. He showed using a different method that the A-property is necessary even for uniqueness with respect to all continuous weights $\omega \in W_C$. However, a price had to be paid in that $U$ should satisfy

\[ \mu(Z(\mu)) = \mu(\text{Int}(Z(\mu))), \quad u \in U. \]

(Here and in what follows $Z(u) = \{x \in K: u(x) = 0\}$, $\mu(S)$ denotes the Lebesgue measure of $S$.) This left open the question whether the A-property is necessary for uniqueness for every $\omega \in W_C$ without any assumption on $U$. In this note we give an affirmative answer to this question showing that restriction (2) can be removed.

**THEOREM 1.** If $U$ is a Chebyshev subspace of $C_\omega(K)$ for every $\omega \in W_C$ then $U$ is an A-space.

In the proof of Theorem 1 we shall follow the approach used in [1] with some more technical details needed for construction of continuous weight.

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The above results indicate that $A$-spaces are the only subspaces of $C(K)$ which guarantee uniqueness of best approximation in $C_\omega(K)$ for every $\omega \in W_C$. On the other hand it will be shown below that each finite-dimensional subspace of $C(K)$ is Chebyshev in $C_\omega(K)$ for some $\omega \in W_C$.

**Theorem 2.** For any finite-dimensional subspace $U$ in $C(K)$ there exists an $\omega \in W_C$ such that $U$ is a Chebyshev subspace of $C_\omega(K)$.

First of all let us recall the definition of $A$-spaces. Denote $U^* = \{u^* \in C(K) : \text{there exists } u \in U \text{ such that } |u| = |u^*| \text{ on } K \}$.

**Definition.** The finite-dimensional subspace $U$ in $C(K)$ is called an $A$-space if for every $u^* \in U^\star \setminus \{0\}$ there exists a $u \in U \setminus \{0\}$ such that $uu^* \geq 0$ on $K$ and $u = 0$ a.e. on $Z(u^*)$.

We shall use in our considerations the following result of Strauss [4].

**Lemma 1.** $U$ is a Chebyshev subspace of $C_\omega(K)$ ($\omega \in W$) if and only if for every $u^* \in U^\star \setminus \{0\}$ there exists a $u \in U$ such that

$$\int_{K \setminus Z(u^*)} \omega u \operatorname{sgn} u^* > \int_{Z(u^*)} \omega |u|.$$ 

In order to prove Theorem 1 we need some further lemmas. As usual $\|g\|_{C(D)}$ denotes supremum norm on $D \subset \mathbb{R}^n$.

Let $D \subset \mathbb{R}^n$ be open and bounded, $M > 0$. Set $W_M(D) = \{\omega \in C(\overline{D}) : \omega > 0 \text{ on } \overline{D}, \omega = M \text{ on } \partial D \text{ and } \int_D \omega \leq 1\}$.

**Lemma 2.** Let $g \in C(D)$, $\|g\|_{C(D)} < \infty$, $g \geq 0$ on $D$ and $\int_D g < 1$. For any $\varepsilon > 0$ there exists $\omega \in W_M(D)$ such that $\|g - \omega\|_{L_1(D)} < \varepsilon$.

**Proof.** Without loss of generality we may assume that $g > 0$ on $D$. Denote $\rho(x) = \text{dist}(x, \partial D)$ and set

$$\Psi_\varepsilon(x) = \begin{cases} 1 - \rho(x)/\varepsilon, & \text{if } \rho(x) < \varepsilon, \\ 0, & \text{if } \rho(x) \geq \varepsilon, \end{cases} \quad \varphi_\varepsilon(x) = \begin{cases} \rho(x)/\varepsilon, & \text{if } \rho(x) < \varepsilon, \\ 1, & \text{if } \rho(x) \geq \varepsilon. \end{cases}$$

Evidently, $\Psi_\varepsilon, \varphi_\varepsilon \in C(\overline{D})$, $\Psi_\varepsilon, \varphi_\varepsilon \geq 0$ on $\overline{D}$. Consider the function $g_\varepsilon(x) = M\Psi_\varepsilon(x) + g(\varphi_\varepsilon(x))$. Then $g_\varepsilon \in C(D)$, $g_\varepsilon > 0$ on $\overline{D}$ and $g_\varepsilon = M$ on $\partial D$. Furthermore, since $g_\varepsilon = g$ if $\rho(x) \geq \varepsilon$ we have

$$\|g - g_\varepsilon\|_{L_1(D)} = \int_{\{x \in D : \rho(x) < \varepsilon\}} |g - g_\varepsilon| \leq (M + 2\|g\|_{C(D)}) \cdot \mu\{x \in D : \rho(x) < \varepsilon\} \to 0 \quad (\varepsilon \to 0).$$

Thus, in particular, $\int_D g_\varepsilon \to \int_D g < 1$ ($\varepsilon \to 0$). This implies that, for $\varepsilon$ sufficiently small, $\int_D g_\varepsilon \leq 1$ and, consequently, $g_\varepsilon \in W_M(D)$. This completes the proof of the lemma.

**Lemma 3.** Let $g \in C(D)$, $\|g\|_{C(D)} < \infty$ and assume that $\int_D g \omega \geq 0$ for every $\omega \in W_M(D)$. Then $g \geq 0$ on $D$.

**Proof.** Let $\tilde{g}(x)$ be equal to 0 if $g(x) \geq 0$ and $\tilde{g}(x) = -g(x)$ if $g(x) < 0$. Obviously, $\tilde{g} \in C(D)$, $\|\tilde{g}\|_{C(D)} < \infty$ and $\tilde{g} \geq 0$ on $D$. We may assume that $\int_D |g| < 1$ which implies that $\int_D \tilde{g} < 1$. Thus we can apply Lemma 1 to $\tilde{g}$; that is
we can find an \( \tilde{\omega} \in W_M(D) \) such that \( \|\tilde{g} - \tilde{\omega}\|_{L_1(D)} < \varepsilon \). Using that \( \int_D \tilde{g} \geq 0 \) and setting \( D_1 = \{x \in D : g(x) < 0\} \), we obtain
\[
\int_{D_1} g^2 \leq \int_{D_1} |g\tilde{\omega} + g^2| - \int_{D_1} g\tilde{\omega} = \int_{D_1} |g\tilde{\omega} + g^2| - \int_D g\tilde{\omega} + \int_{D\setminus D_1} g\tilde{\omega}
\]
\[
\leq \int_{D_1} |g\tilde{\omega} + g^2| + \int_{D\setminus D_1} |g|\tilde{\omega} \leq \|g\|_{C(D)} \left\{ \int_{D_1} |\tilde{\omega} + g| + \int_{D\setminus D_1} \tilde{\omega} \right\}
\]
\[
= \|g\|_{C(D)} \|\tilde{\omega} - \tilde{g}\|_{L_1(D)} < \varepsilon \|g\|_{C(D)}.
\]
Since \( \varepsilon > 0 \) is arbitrary this yields that \( \int_{D_1} g^2 = 0 \), i.e. \( \mu(D_1) = 0 \). This and continuity of \( g \) on \( D \) imply that \( g \geq 0 \) on \( D \).

**Proof of Theorem 1.** Consider an arbitrary \( u^* \in U^* \setminus \{0\} \). By Lemma 1 for every \( \omega \in W_C \) there exists a \( u \in U \) such that
\[
\int_{K \setminus Z(u^*)} \omega u \operatorname{sgn} u^* > \int_{Z(u^*)} \omega |u|.
\]
Set \( \tilde{U} = \{u \in U : u = 0 \text{ a.e. on } Z(u^*)\} \), and let \( U = \tilde{U} \oplus U_1 \). This means that \( U_1 \) is a linear subspace of \( U \) and elements of \( U_1 \) cannot vanish a.e. on \( Z(u^*) \). Hence there exists a constant \( M > 0 \) such that
\[
\|u_1\|_{C(K)} \leq M \int_{Z(u^*)} |u_1| \tag{4}
\]
holds for every \( u_1 \in U_1 \). Set \( D = K \setminus Z(u^*) \) and consider the set \( W_M(D) \). Let \( \varphi_1, \ldots, \varphi_r \) be a basis in \( \tilde{U} \). We introduce the set
\[
A_r = \left\{ \left( \int_D \omega \varphi_i \operatorname{sgn} u^* \right)_{i=1}^r : \omega \in W_M(D) \right\}.
\]
Evidently, \( A_r \) is a convex set in \( R^r \). Furthermore, Lemma 2 implies that \( 0 \in \overline{A_r} \). We claim that \( 0 \notin A_r \). Assume that in contradiction \( 0 \in A_r \). Then for some \( \tilde{\omega} \in W_M(D) \)
\[
\int_D \tilde{\omega} u \operatorname{sgn} u^* = 0, \quad u \in \tilde{U}.
\]
Extend \( \tilde{\omega} \) to \( Z(u^*) \) setting \( \tilde{\omega} = M \) on \( Z(u^*) \). Obviously, \( \tilde{\omega} \in W_C \). Consider an arbitrary \( u = u_1 + \tilde{u} \in U \), where \( u_1 \in U_1, \tilde{u} \in \tilde{U} \). Then by (5) and (4)
\[
\left| \int_D \tilde{\omega} u \operatorname{sgn} u^* \right| = \left| \int_D \tilde{\omega} u_1 \operatorname{sgn} u^* \right| 
\]
\[
\leq \|u_1\|_{C(K)} \int_D \tilde{\omega} \leq \|u_1\|_{C(K)} \leq M \int_{Z(u^*)} |u_1| = \int_{Z(u^*)} \tilde{\omega} |u|.
\]
But this contradicts (3). Hence \( 0 \notin A_r \), i.e. \( 0 \in \text{Bd} A_r \). Then there exists a hyperplane supporting \( A_r \) at \( 0 \); that is, for some \( u \in \tilde{U} \setminus \{0\} \) we have that
\[
\int_D \omega u \operatorname{sgn} u^* \geq 0
\]
for every \( \omega \in W_M(D) \). This and Lemma 3 imply that \( uu^* \geq 0 \) on \( D \), and therefore \( uu^* \geq 0 \) on \( K \), as well. Moreover, \( u = 0 \text{ a.e. at } Z(u^*) \), and thus the \( A \)-property of \( U \) is verified.
PROOF OF THEOREM 2. Let \( U = \text{span}\{u_1, \ldots, u_n\} \). Since \( K = \text{Int} K \) it follows that \( u_1, \ldots, u_n \) are linearly independent on \( \text{Int} K \). Therefore, there exist points \( x_1, \ldots, x_n \in \text{Int} K \) such that
\[
\det \begin{pmatrix}
    u_1(x_1), & \ldots, & u_1(x_n) \\
    \vdots & & \vdots \\
    u_n(x_1), & \ldots, & u_n(x_n)
\end{pmatrix} \neq 0.
\]
(6)

Denote by \( B_r(x) \) the closed ball with center at \( x \) and radius \( r \). Set
\[
D_1^{(m)} = \bigcup_{i=1}^{n} B_{1/m}(x_i); \quad D_2^{(m)} = \bigcup_{i=1}^{n} B_{1/m+1/m^2}(x_i).
\]
For a suitable \( m_0 \) we have that \( D_1^{(m)} \subseteq D_2^{(m)} \subseteq \text{Int} K \) if \( m \geq m_0 \). Consider a weight \( \omega_m \) equal to \( 1 \) and \( 1/m^{s+1} \) on \( D_1^{(m)} \) and \( K \setminus D_2^{(m)} \), respectively. We can easily extend \( \omega_m \) to \( D_2^{(m)} \setminus D_1^{(m)} \) in such a way that \( \omega_m \in W_C \) and \( \omega_m \leq 1 \) on \( K \). Assume that the statement of the theorem is false. Then, in particular, \( U \) is not Chebyshev in \( C_{\omega_m}(K) \) for every \( m \geq m_0 \). It follows from Lemma 1 that there exists \( u_m^* \in U^* \setminus \{0\} \), \( \|u_m^*\|_{C(K)} = 1 \) such that, for every \( u \in U \),
\[
\left| \int_{K \setminus Z(u_m^*)} \omega_m u \text{sgn } u_m^* \right| \leq \int_{Z(u_m^*)} \omega_m|u|, \quad (m \geq m_0).
\]
(7)

By definition of \( U^* \) there exists \( u_m \in U \) satisfying \( |u_m^*| = |u_m| \) on \( K \). In particular, we obtain that \( \|u_m\|_{C(K)} = 1 \). Therefore, without loss of generality we may assume that \( u_m \rightarrow u_0 \in U \) in \( C(K) \)-norm, \( \|u_0\|_{C(K)} = 1 \). By (6), \( u_0(x_r) \neq 0 \) for at least one \( 1 \leq r \leq n \). Moreover, (6) also implies that there exists \( \tilde{u} \in U \setminus \{0\} \) such that \( \tilde{u}(x_r) = 1 \) and \( \tilde{u}(x_i) = 0 \) for \( 1 \leq i \leq n, i \neq r \). Applying (7) to \( \tilde{u} \) we have
\[
\left| \int_{K \setminus Z(u_m^*)} \omega_m \tilde{u} \text{sgn } u_m^* \right| \leq \int_{Z(u_m^*)} \omega_m|\tilde{u}|, \quad m \geq m_0.
\]
(8)

Since \( u_0(x_r) \neq 0 \) and \( |u_m| = |u_m| \rightarrow |u_0| \) \( (m \rightarrow \infty) \) uniformly on \( K \) it follows that \( B_{1/m}(x_r) \subseteq K \setminus Z(u_m^*) \) for \( m \geq m_1 \) with sufficiently large \( m_1 \). Therefore, we obtain by (8)
\[
\left| \int_{B_{1/m}(x_r)} \tilde{u} \right| = \int_{B_{1/m}(x_r)} \omega_m \tilde{u} \text{sgn } u_m^* \leq \int_{K \setminus B_{1/m}(x_r)} \omega_m|\tilde{u}|
\]
(9)
\[
= \sum_{i=1}^{n} \int_{B_{1/m}(x_i)} \omega_m|\tilde{u}| + \int_{D_2^{(m)} \setminus D_1^{(m)}} \omega_m|\tilde{u}| + \int_{K \setminus D_2^{(m)}} \omega_m|\tilde{u}|.
\]

Furthermore, we have
\[
\int_{K \setminus D_2^{(m)}} \omega_m|\tilde{u}| = \frac{1}{m^{s+1}} \int_{K \setminus D_2^{(m)}} |\tilde{u}| \leq \frac{\mu(K)\|\tilde{u}\|_{C(K)}}{m^{s+1}},
\]
(10)
\[
\int_{D_2^{(m)} \setminus D_1^{(m)}} \omega_m|\tilde{u}| \leq \int_{D_2^{(m)} \setminus D_1^{(m)}} |\tilde{u}| \leq \|\tilde{u}\|_{C(K)} \mu(D_2^{(m)} \setminus D_1^{(m)}) \leq \frac{a_s\|\tilde{u}\|_{C(K)}}{m^{s+1}}
\]
(11)
with a constant \( a_s > 0 \) depending only on \( s \). Finally, using that \( \tilde{u}(x_i) = 0 \) for \( 1 \leq i \leq n, i \neq r \), we obtain

\[
\sum_{i=1}^{n} \int_{B_{1/m}(x_i)} \omega_m |\tilde{u}| = \sum_{i=1}^{n} \int_{B_{1/m}(x_i)} |\tilde{u}| \leq \frac{C_s(n-1)\omega(\tilde{u}, 1/m)}{m^s},
\]

where \( C_s > 0 \) depends only on \( s \) and \( \omega(\tilde{u}, h) \) is the uniform modulus of continuity of \( \tilde{u} \). Applying inequalities (10)–(12) in (9) leads to

\[
\left| \int_{B_{1/m}(x_r)} \tilde{u} \right| \leq o(m^{-s}), \quad m \geq m_1,
\]

which in turn implies that \( \tilde{u}(x_r) = 0 \), a contradiction. The theorem is proved.

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