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A REMARK ON SINGULAR CALDERÓN-ZYGMUND THEORY

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ABSTRACT. It is shown that in $\mathbf{R}^n$ the operator

$$Hf(x) = pv \int_{-\infty}^{+\infty} f(x_1 - t, \dots x_n - t^n) t^{-1}\, dt$$

maps $L(\log L)$ to weak L^1 locally. A slight variant of the Calderón-Zygmund procedure provides a new approach to the previously known L^p boundedness of H, $1 < p < \infty$. Relatively sharp bounds are obtained as $p \to 1^+$, and extrapolation produces the result for $L(\log L)$.

Let $a = a_1 < \cdots < a_n \in \mathbf{Z}$, and for all $t \in \mathbf{R}$ let $\gamma(t) = (t^{a_1}, \dots, t^{a_n}) \in \mathbf{R}^n$. The L^p boundedness of operators such as $Hf(x) = pv \int f(x - \gamma(t))t^{-1}\, dt$ has been studied by a number of authors [**SW**, **JRdF**, **G**, **8**, **C1**, **C2**, **PS**]. In all this work there are two main steps: First one proves L^2 bounds, and then further arguments are used to pass to L^p. A prototypical technique for deducing L^p bounds from L^2 bounds is the Calderón-Zygmund theory of singular integrals. As is well known [**S1**], if K agrees with a function away from the origin, $\hat{K} \in L^\infty$ and K satisfies the Hörmander condition (1) below, then convolution with K is bounded on all L^p and is of weak type on L^1. H is a limiting case just outside the scope of that theory, for it is given by convolution with a distribution K which is homogeneous under the family of dilations $\delta_r x = (r^{a_1} x_1, \dots, r^{a_n} x_n)$, but which is equal to a difference of two Dirac measures when restricted to the unit sphere. The Hörmander condition fails, but several substitute arguments have been found [**SW**, **JRdF**, **C1**]. Our purpose here is to indicate a variant of the Calderón-Zygmund procedure which does apply to H; the main point will be that H satisfies a certain generalization of the Hörmander condition. Our method applies to a more general class of convolution operators as well as to related maximal functions.

This technique appears to be slightly more precise than the reasoning used in previous studies of H and related operators. Fix a bounded subset B of $\mathbf{R}^n$. $L^{1,\infty}$ denotes the usual space weak L^1, equipped with the natural quasi-norm.

THEOREM 1. *H is a bounded operator from $L(\log L)(B)$ to $L^{1,\infty}(B)$. The same holds for the maximal function $M_\gamma f(x) = \sup_{r>0} r^{-1} \int_{0<t<r} |f(x - \gamma(t))|\, dt$.*

COROLLARY. *$\lim_{r\to 0} r^{-1} \int_{0<t<r} f(x - \gamma(t))\, dt = f(x)$ a.e. for all f locally in $L(\log L)$.*

Define $|x|$ to be the reciprocal of that value of r for which $\delta_r x \in S^{n-1}$ if $x \neq 0$, and $|0| = 0$. Then a triangle inequality $|x + y| \leq C_0(|x| + |y|)$ holds. By a ball we

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shall mean an open set $B = B(x,r) = \{y : |x-y| < r\}$, the double B^* of B is defined to be $B(x, 2C_0 r)$, and B^{**} is the double of B^*. φ denotes a C^∞ function supported in $B(0,1)$ satisfying $\int \varphi = 1$. $\varphi_j(x) = r^{-a}\varphi(\delta_{r^{-1}}x)$ where $r = 2^j$ and $a = \sum a_i$. Let ψ be C^∞, be identically one in $B(0,1)$, and be supported in $B(0,2)$, and let $\varsigma_j(x) = \psi(\delta_{2r}x) - \psi(\delta_r x)$ where $r = 2^{-j-1}$. Finally for any tempered distribution k we define $k_j = \varsigma_j k$. (The symbols k, φ, ς will be used in a consistent manner, so the duplication of notation need cause no confusion.) $f_{(z)}$ denotes the translate $f_{(z)}(x) = f(x+z)$, and the translate $k^{(z)}$ of a tempered distribution is defined by duality: $\langle f, k^{(z)}\rangle = \langle f_{(z)}, k\rangle$. The L^p multiplier norm of a tempered distribution is $|k|_p \equiv \sup_{f \in L^p} \|f^*k\|_p / \|f\|_p$, where $\|f\|_p$ is the usual L^p norm with respect to Lebesgue measure. We also write $|T|_p$ for the operator norm of an operator T, and denote the adjoint of T by T^*. When $Tf \equiv f^*k$ and $k \in L^1$, $T_j f$ is defined to be f^*k_j.

T is said to satisfy the Hörmander condition if it is given by convolution with a distribution which coincides away from the origin with a function k satisfying

$$\|k - k^{(y)}\|_{L^1\{x:|x|>r\}} < C \quad \text{for all } |y| < \eta r, \text{ for all } r > 0 \tag{1}$$

for some constants C and η. Operators which have arisen in practice generally satisfy a slightly stronger condition:

$$\|k - k^{(y)}\|_{L^1\{x:|x|>\rho r\}} < C\rho^{-\varepsilon} \quad \text{for all } |y| < \eta r \text{ and all } r > 0, \rho > 1 \tag{2}$$

for some finite positive C, η, ε. Our substitute will be

$$|k_{j+i} - k_{j+i}^{(y)}|_p < C2^{-\varepsilon i} \quad \text{for all } |y| < \eta 2^j \text{ and all } j \in \mathbf{Z}, i \in \mathbf{Z}^+ \tag{3_p}$$

for some C, η, ε. Note that when $p = 1$ the L^1 and $|\cdot|_1$ norms coincide, and (3_p) is essentially equivalent to (2). Many operators, including those considered in the standard Calderón-Zygmund theory, satisfy

$$\|k_j\|_{L^1} < C. \tag{4}$$

When (4) holds, (3_2) implies (3_p) for all $1 < p < 2$ by interpolation; conversely (3_p) is equivalent to (3_q) where $q = p'$, and hence (3_p) always implies (3_2) for any p.

THEOREM 2. *Suppose that $Tf = f^*k$ where k is a finite measure. Suppose that*

$$|T_i T_j^*|_2 + |T_i^* T_j|_2 < C2^{-\varepsilon|i-j|} \tag{5}$$

for all i, j, for some $\varepsilon > 0$. Suppose that (4) *and* (3_2) *hold. Then T is bounded on L^p for all $1 < p < \infty$, with an operator norm depending only on the constants in* (3_2), (4) *and* (5). *Moreover, T is a bounded operator from $L(\log L)(B)$ to $L^{1,\infty}(B)$ for any bounded set B.*

For truncated Hilbert transforms $H_{ab}f(x) = \int_{a<|t|<b} f(x - \gamma(t))t^{-1}\,dt$ with γ as above, well-known bounds on Fourier transforms [**SW**] imply (3_2) and (5), and therefore this theorem implies the L^p boundedness of H_{ab} uniformly in a, b, and hence the uniform bound $\|Hf\|_p \le C\|f\|_p$ for all f in the dense subspace C_0^1 of L^p.

Our proof requires a parabolic Calderón-Zygmund decomposition whose proof is well known [**CW**]:

LEMMA 1. *For any* $f \in L^p$ *and* $\alpha > 0$ *there exist* g *and* $\{b_i\}$ *such that* $f = g + \sum b_i$, $\|g\|_\infty < \alpha$, *each* b_i *is supported on a ball* $B_i = B(x_i, 2^{j(i)})$, *the supports of the* b_i *are pairwise disjoint,* $\sum |B_i| \le C\alpha^{-p}\|f\|_p^p$, $|B_i|^{-1}\int_B |b_i|^p \le C\alpha^p$, *and no point of* $\mathbf{R}^n$ *is contained in more than* C *of the doubles* B_i^*.

LEMMA 2. *Let* $\{S_j\}$ *be a finite set of bounded operators on a Hilbert space, and let* $|T|$ *denote the operator norm of an operator* T *and* T^* *its adjoint. Suppose that* $|S_i^* S_j| + |S_i S_j^*| \le C2^{-\varepsilon|i-j|}$ *for all* i, j. *Then there exists a constant* B *depending on* C *and on* ε, *but not on the number of* S_j, *such that* $|\sum S_j| \le C$.

Lemma 2 is due to Cotlar and Stein. At one point the classical proof of the weak-type (1,1) bound, the fact that $\sum \|b_i\|_{L^1} = \|\sum b_i\|_{L^1} \le \|f\|_{L^1}$ is used; unfortunately the equality fails for L^p, $p > 1$. Instead we shall use

LEMMA 3. *Suppose that operators* $\{\Upsilon_j\}$ *satisfy* $|\Upsilon_j|_1 \le A_1$ *and* (5). *Let* $1 < p \le 2$. *Then there exists* $D < \infty$ *such that, for any sequence* $\{f_j\} \subset L^p$, $\|\sum \Upsilon_j f_j\|_p \le D(\sum \|f_j\|_p^p)^{1/p}$.

PROOF. The case $p = 1$ follows at once from (4) and the triangle inequality. For $p = 2$, consider the operator $F = \{f_j\} \to \sum \Upsilon_j f_j = \Upsilon F$ from $L^2(\ell^2)$ to L^2. Set $S_j F = \Upsilon_j f_j$. Lemma 2, applied to the decomposition $\Upsilon = \sum S_j$, implies by (5) that Υ is bounded. Intervening values of p are treated by complex interpolation; therefore $D \le A_1^{1-\theta} A_2^\theta$ where A_2 is the operator norm of Υ from $L^2(\ell^2)$ to L^2 and $0 < \theta = \theta(p) \le 1$.

To prove Theorem 2, note that (5) implies the L^2 boundedness of T by Lemma 2. Hence by duality and the Marcinkiewicz interpolation theorem, it suffices to show that T is of weak type on L^p for each $1 < p < 2$. Fix such a p, and an $\alpha > 0$. Given $f \in L^p$, construct g and $\{b_i\}$ as in Lemma 1. Set $G = g + \sum(b_i * \varphi_{j(i)})$; then $\|G\|_\infty \le C\alpha$ since $\|b_i * \varphi_{j(i)}\|_\infty \le C\alpha$, $b_i * \varphi_{j(i)}$ is supported on B_i^* by the triangle inequality, and the B_i^* have the bounded overlap property. Similarly, $\|G\|_p \le C\|f\|_p$, so the L^2 boundedness of T together with Chebychev's inequality gives $|\{x : |TG(x)| > \alpha\}| \le C\alpha^{-p}\|f\|_p^p$. Thus it suffices to treat

$$\begin{aligned}\sum Tb_i - \sum T(b_i * \varphi_{j(i)}) &\equiv \sum b_i * k * (\delta - \varphi_{j(i)}) \\ &= \sum b_i * \left[\sum_{s>0} k_{s+j(i)} * (\delta - \varphi_{j(i)})\right] \\ &\quad + \sum b_i * \left[(k \cdot \psi_{j(i)}) * (\delta - \varphi_{j(i)})\right] = I + II.\end{aligned}$$

II is supported on $\bigcup B_i^{**}$, a set whose measure is at most $\sum |B_i^{**}| \le C\sum |B_i| \le C\alpha^{-p}\|f\|_p^p$. Then it suffices to show that $\|I\|_{L^p(\mathbf{R}^n/\cup B^{**})} \le C\|f\|_p$; in fact we shall establish the same bound for $\|I\|_{L^p(\mathbf{R}^n)}$.

The smoothness hypothesis (3_2) implies

$$|k_{j+s} * (\delta - \varphi_j)|_2 \le C2^{-\varepsilon s} \quad \text{for all } j \in \mathbf{Z}, s \in \mathbf{Z}^+. \tag{6}$$

Let $a_j = \sum_{j(i)=j} b_i$. Then $I = \sum_{s>0}[\sum_j a_j * k_{j+s} * (\delta - \varphi_j)]$. Fix s and set $\Upsilon_j f = f * k_{j+s} * (\delta - \varphi_j)$. (5), (6) and Lemma 2 imply that the operator norm of Υ from $L^2(\ell^2)$ to L^2 is $\le C2^{-\sigma s}$ for some $\sigma > 0$ ($\sigma < \varepsilon/2$). Therefore, $\|\sum \Upsilon_j a_j\|_p \le C2^{-\theta\sigma s}(\sum \|a_j\|_p^p)^{1/p} \le C2^{-\theta\sigma s}\|f\|_p$ for some $0 < \theta < 1$, and summing over s concludes the proof.

The result for $L(\log L)$ is proved by extrapolating the bounds obtained from the above argument. Suppose that $f \in L(\log L)(B)$ and $\alpha > 0$ are given. Apply Lemma 1 with $p = 1$. Then the measure of the set on which $|TG|$ is greater than $\alpha/2$ is at most $C\alpha^{-1}\|f\|_{L^1} \leq C\alpha^{-1}\|f\|_{L(\log L)}$. Let E be the union of the B_i^{**}. Then $|E| \leq C\alpha^{-1}\|f\|_{L^1} \leq C\alpha^{-1}\|f\|_{L(\log L)(B)}$ since B is bounded. Hence it suffices to show

$$\|I\|_{L^1(B)} \leq C\left\|\sum b_i\right\|_{L(\log L)}$$

with a bound C independent of α and of the balls B_i. For all p, $\|\sum b_i\|_{L^p} \leq C\|f\|_{L^p}$ with C independent of p. The above argument gives $\|I\|_{L^p} \leq C(p-1)^{-1}\|\sum b_i\|_{L^p}$ with C independent of p for all $1 < p \leq 2$. Consider the linear operator $\{b_i\} \to I$ from $L^p(\bigcup B_i)$ to $L^p(\mathbf{R}^n)$. By the extrapolation theorem of Yano [**Y**] it is bounded from $L(\log L)(\bigcup B_i)$ to $L^1(B)$.

The maximal function M_γ is treated by introducing a square function and applying the same reasoning to it. Fix a nonnegative C^∞ auxiliary function ς supported on $(\frac{1}{4}, 2)$ and positive on $[\frac{1}{2}, 1]$. Define measures μ_j on $\mathbf{R}^n$ by $\langle f, \mu_j\rangle = \int f(\gamma(t))\varsigma(2^{-j}t)2^{-j}\,dt$. Then for all nonnegative f there is a pointwise bound $M_\gamma f(x) \leq C\sup_j |f * \mu_j(x)|$. To bound the latter maximal function we proceed as for H. Since it is certainly bounded on L^∞ matters reduce to the estimation of the analogue of I, the sum over all integers $s > 0$ of the square functions $S_s(x) = (\sum_{m\in\mathbf{Z}} |\sum_{i:j(i)=m} b_i * (\delta - \varphi_{j(i)}) * \mu_{j(i)+s}(x)|^2)^{1/2}$ (for more details see [**C4**]). The L^2 bound $\|S_s\|_{L^2}^2 \leq C2^{-\varepsilon s}\sum\|b_i\|_{L^2}^2$ follows as before. The easy L^1 bound $\|S_s\|_{L^1} \leq C\sum\|b_i\|_{L^1} \leq C\|f\|_{L^1}$ is obtained by dominating the ℓ^2 norm in the definition of S_s by the ℓ^1 norm and applying the triangle inequality.

Analogues of the proposition hold in greater generality, for instance on homogeneous nilpotent Lie groups. Thus we have another proof of the L^p results of [**C1**], and in fact an extension of the differentiation results of [**C1** and **C3**] to $L(\log L)$ since the appropriate analogues of the L^2 hypotheses (3_2) and (5) are verified there. However, the method does not apply to those curves treated in [**8**], which may contain line segments and hence lack curvature. Moreover, hypothesis (4) fails to hold for certain more singular variants considered in [**SW**] implicitly and [**G**] explicitly, where even away from the origin k is a distribution of positive order.

It is not known whether operators of the types considered here are of weak type (1,1). A weaker conjecture would be that they map $L(\log L)(B)$ to $L^1(B)$, rather than merely to weak L^1.

In dimension $n = 2$ results more refined than those in this note have already been obtained, for the maximal function M_γ, in [**C4**].

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