PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 99, Number 1, January 1987

THE FIXED POINTS OF AN ANALYTIC SELF-MAPPING

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ABSTRACT. Let R be a hyperbolic Riemann surface embedded in a compact Riemann surface of genus g and let f be an analytic function mapping R into R, f not the identity function. Then f has as most $2g+2$ distinct fixed points in R; equality may hold. If f has 2 or more distinct fixed points, then f is a periodic conformal automorphism of R onto itself. This paper contains a proof of this theorem and several related results.

This paper contains a proof of the following theorem.

THEOREM. *Let R be a hyperbolic Riemann surface embedded in a compact Riemann surface of genus g and let f be an analytic function mapping R into R, f not the identity function. Then f has at most $2g+2$ distinct fixed points in R; equality may hold. If f has 2 or more distinct fixed points, then f is a periodic conformal automorphism of R onto itself.*

COROLLARY. *Let Ω be a domain in the complex plane with at least two (finite) boundary points and let f be an analytic function mapping Ω into itself, f not the identity function. Then f has at most two distinct fixed points.*

Some comments on this theorem and its proof will be found at the end of the paper.

We begin with a theorem that is independent of genus.

THEOREM 1. *Let R be a hyperbolic Riemann surface which is not conformally equivalent to a disc and let f be a nonconstant analytic function mapping R into R. If f has two or more distinct fixed points in R, then f is a periodic conformal automorphism of R onto R.*

PROOF. Let z_0 be a fixed point of f in R; let T be the uniformizer of R with $T(0) = z_0$. There is an analytic function g mapping Δ into Δ with $T \circ g = f \circ T$. Hence, $T(g(0)) = f(T(0)) = F(z_0) = z_0$ so that there is an element γ of the group of deck transformations with $\gamma(g(0)) = 0$. Let $g_1 = \gamma \circ g$. Then $g_1(0) = 0$ and $T \circ g_1 = T \circ \gamma \circ g = T \circ g = f \circ T$. Hence, we may initially assume that $g(0) = 0$.

We define the iterates of f by

$$f_1 = f \quad \text{and} \quad f_{n+1} = f \circ f_n, \qquad n = 1, 2, \ldots.$$

Similarly, we define $g_1, g_2, \ldots$, the iterates of g. It follows that

$$T \circ g_n = f_n \circ T, \qquad n = 1, 2, \ldots.$$

The functions $\{g_n\}$ form a normal family.

Received by the editors May 6, 1985 and, in revised form, December 11, 1985.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30C20, 55M20; Secondary 30F99, 57M12, 54H25.

Research of both authors supported in part by the National Science Foundation.

Let $S = T^{-1}(z_0)$; because T is a covering map S is a discrete subset of Δ. Note that S is not a singleton since R is not conformally a disc. If $w \in S$, then $T(g(w)) = f(T(w)) = f(z_0) = z_0$ so that g maps S into itself.

If $|g'(0)| < 1$, then the sequence $\{g_n\}$ converges uniformly on compact subsets of Δ to the function which is identically zero and hence $f_n(p) \to z_0$ as $n \to \infty$ for each $p \in R$. In this case f has just one fixed point. This shows that it must be the case that $|g'(0)| = 1$. Consequently, $g(z) = \lambda_0 z$, $\lambda_0 = \exp(2\pi i\theta_0)$ for a unique $\theta_0 \in [0, 1)$. We shall show that θ_0 is rational. Suppose to the contrary that θ_0 is irrational. Let w be any point in S, $w \neq 0$. Since g is a rotation about 0 of $2\pi\theta_0$ radians it follows that $S \cap \{|z| \leq r\}$ is not discrete for some $r < 1$. This contradicts the fact that T is a covering map. Hence, $\theta_0 = M/N$ where M and N are positive integers with no common factors. Consequently, g_N is the identity and so f_N is also the identity. This then implies that f is a one-to-one mapping of Ω onto itself.

REMARK. It is elementary to see that an analytic function mapping Δ into Δ has at most 1 (distinct) fixed point unless it is the identity. Thus, the exclusion of the disc in Theorem 1 is not important.

PROOF OF THE THEOREM. There is no loss in assuming initially that f has at least two fixed points. Applying Theorem 1, we see that f is a periodic conformal automorphism of R onto itself, say $f_N =$ identity.

Let u be a continuous function on the closure of R which is zero on the boundary of R and positive and C^∞ on R. Set

$$H(z) = \begin{cases} \sum_{k=1}^{N} u(f_k(z)), & z \in R, \\ 0, & z \in \partial R. \end{cases}$$

H is a continuous function on $R \cup \partial R$ which is positive and C^∞ on R. Let E be the critical set of H on R and let $\{\varepsilon_n\}$ be a sequence of positive numbers decreasing to zero with no ε_n contained in $H(E)$. Such numbers exist by Sard's theorem. Define R_n to be that component of the set $H^{-1}(\varepsilon_n, \infty)$ which contains the point z_0. Then $R_1 \subset R_2 \subset \cdots$, the union of all the R_n is R, and f maps R_n onto itself for each n since $H \circ f = H$. Further, by the inverse function theorem, ∂R_n is the union of a finite number of disjoint, smooth, simple closed curves. Finally, the genus of each R_n is no more than g since the genus of a surface is the maximal number of simple closed curves that may be deleted from the surface without disconnecting it. We shall show that f has no more than $2 + 2g$ fixed points in any R_n; this will complete the proof of the theorem.

Fix n and let S denote R_n. Let S' denote the surface obtained by identifying the orbits $\{f_k(z): k = 1, 2, \ldots, N\}$ to a point and let F be the quotient map from S onto S'. We may now apply the Riemann-Hurwitz relation [**1**, Theorem I.2.7] to F, S, and S':

$$2g = 2d(g' - 1) + 2 + B$$

where g is the genus of S, g' the genus of S', d the degree of F, and B the total branching number of F. In this case, we know that $d = N$ and $B = (N-1)\alpha$ where α is the number of fixed points of f. This yields

$$(N-1)\alpha = 2g - 2N(g' - 1) - 2 \leq 2g + 2N - 2.$$

Consequently,

$$\alpha \leq 2 + 2g/(N-1) \leq 2 + 2g.$$

This completes the proof of the theorem.

EXAMPLES. 1. The function $g(z) = ze^z$ maps the surface $R = \mathbf{C}\backslash\{0\}$ into itself and has infinitely many fixed points.

2. For $N \geq 2$, the polynomial $p(z) = z^N + z - 1$ maps the sphere onto itself and the plane onto itself and has N distinct fixed points.

3. Let R be the compact Riemann surface of genus g obtained by taking the connected sum of g tori. The mapping f obtained by rotating R by $180°$ is the hyperelliptic involution of R and it has $2g+2$ fixed points. Indeed, the proof of the theorem shows that the number of fixed points on a compact surface can equal $2g+2$ only when $N = 2$; in this case, R must be hyperelliptic and f the unique involution of R; see [**1**, Proposition III.7.9].

REMARKS. 1. The problem of determining the number of fixed points of an analytic function mapping a planar domain into itself was originally brought to our attention by L. A. Rubel.

2. The Theorem has a more direct, but deeper proof. Once Theorem 1 is established, we could appeal to a result of B. Maskit [**3**] which asserts that there is a closed Riemann surface R^* of genus g and a conformal embedding of R into R^* so that, under this embedding, every conformal self-map of R is the restriction of a conformal self-map of R^*. An application of the Riemann-Hurwitz theorem then completes the proof, as it did ours.

3. A version of our Theorem 1 is also to be found in [**2**] as Theorem 4 although the proof is considerably different.

4. The final paragraph of the proof of the Theorem can be modified to prove the following more general result.

PROPOSITION. *Let f be a diffeomorphism of a compact connected surface S, which may have boundary and which has genus g. Suppose that f_N, the Nth iterate of f, is the identity, and f_k is not the identity if $0 < k < N$. If the number α of fixed points of f is finite, then α satisfies $\alpha \leq 2 + 2g/(N-1)$.*

PROOF. By averaging a metric we may asume f is an isometry. Let S' denote the surface obtained by identifying orbits of f to a point and let $F\colon S \to S'$ be the quotient map. Since f is an isometry, in a neighborhood of a fixed point it is a rotation through an angle of $2\pi M/N$. (If it was a reflection there would be infinitely many fixed points.) The integer M must be prime to N since otherwise for some k, $0 < k < N$, f_k is the identity on a neighborhood of the fixed point and hence on all of S. The remainder of the proof now is the same as that of the Theorem. We note that the Riemann-Hurwitz relation holds in this more general context, and the proof in [**1**] is adequate to show this.

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