THE FIXED POINTS OF AN ANALYTIC SELF-MAPPING

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Abstract. Let \( R \) be a hyperbolic Riemann surface embedded in a compact Riemann surface of genus \( g \) and let \( f \) be an analytic function mapping \( R \) into \( R \), \( f \) not the identity function. Then \( f \) has as most \( 2g + 2 \) distinct fixed points in \( R \); equality may hold. If \( f \) has 2 or more distinct fixed points, then \( f \) is a periodic conformal automorphism of \( R \) onto itself. This paper contains a proof of this theorem and several related results.

This paper contains a proof of the following theorem.

Theorem. Let \( R \) be a hyperbolic Riemann surface embedded in a compact Riemann surface of genus \( g \) and let \( f \) be an analytic function mapping \( R \) into \( R \), \( f \) not the identity function. Then \( f \) has at most \( 2g + 2 \) distinct fixed points in \( R \); equality may hold. If \( f \) has 2 or more distinct fixed points, then \( f \) is a periodic conformal automorphism of \( R \) onto itself.

Corollary. Let \( \Omega \) be a domain in the complex plane with at least two (finite) boundary points and let \( f \) be an analytic function mapping \( \Omega \) into itself, \( f \) not the identity function. Then \( f \) has at most two distinct fixed points.

Some comments on this theorem and its proof will be found at the end of the paper.

We begin with a theorem that is independent of genus.

Theorem 1. Let \( R \) be a hyperbolic Riemann surface which is not conformally equivalent to a disc and let \( f \) be a nonconstant analytic function mapping \( R \) into \( R \). If \( f \) has two or more distinct fixed points in \( R \), then \( f \) is a periodic conformal automorphism of \( R \) onto \( R \).

Proof. Let \( z_0 \) be a fixed point of \( f \) in \( R \); let \( T \) be the uniformizer of \( R \) with \( T(0) = z_0 \). There is an analytic function \( g \) mapping \( \Delta \) into \( \Delta \) with \( T \circ g = f \circ T \). Hence, \( T(g(0)) = f(T(0)) = F(z_0) = z_0 \) so that there is an element \( \gamma \) of the group of deck transformations with \( \gamma(g(0)) = 0 \). Let \( g_1 = \gamma \circ g \). Then \( g_1(0) = 0 \) and \( T \circ g_1 = T \circ \gamma \circ g = T \circ g = f \circ T \). Hence, we may initially assume that \( g(0) = 0 \).

We define the iterates of \( f \) by

\[ f_1 = f \quad \text{and} \quad f_{n+1} = f \circ f_n, \quad n = 1, 2, \ldots. \]

Similarly, we define \( g_1, g_2, \ldots \), the iterates of \( g \). It follows that

\[ T \circ g_n = f_n \circ T, \quad n = 1, 2, \ldots. \]

The functions \( \{g_n\} \) form a normal family.

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Let $S = T^{-1}(z_0)$; because $T$ is a covering map $S$ is a discrete subset of $\Delta$. Note that $S$ is not a singleton since $R$ is not conformally a disc. If $w \in S$, then $T(g(w)) = f(T(w)) = f(z_0) = z_0$ so that $g$ maps $S$ into itself.

If $|g'(0)| < 1$, then the sequence $\{g_n\}$ converges uniformly on compact subsets of $\Delta$ to the function which is identically zero and hence $f_n(p) \to z_0$ as $n \to \infty$ for each $p \in R$. In this case $f$ has just one fixed point. This shows that it must be the case that $|g'(0)| = 1$. Consequently, $g(z) = \lambda_0 z$, $\lambda_0 = \exp(2\pi i \theta_0)$ for a unique $\theta_0 \in [0,1)$. We shall show that $\theta_0$ is rational. Suppose to the contrary that $\theta_0$ is irrational. Let $w$ be any point in $S$, $w \neq 0$. Since $g$ is a rotation about 0 of $2\pi \theta_0$ radians it follows that $S \cap \{|z| < r\}$ is not discrete for some $r < 1$. This contradicts the fact that $T$ is a covering map. Hence, $\theta_0 = M/N$ where $M$ and $N$ are positive integers with no common factors. Consequently, $g_N$ is the identity and so $f_N$ is also the identity. This then implies that $f$ is a one-to-one mapping of $\Omega$ onto itself.

**Remark.** It is elementary to see that an analytic function mapping $\Delta$ into $\Delta$ has at most 1 (distinct) fixed point unless it is the identity. Thus, the exclusion of the disc in Theorem 1 is not important.

**Proof of the Theorem.** There is no loss in assuming initially that $f$ has at least two fixed points. Applying Theorem 1, we see that $f$ is a periodic conformal automorphism of $R$ onto itself, say $f_N = \text{identity}$.

Let $u$ be a continuous function on the closure of $R$ which is zero on the boundary of $R$ and positive and $C^\infty$ on $R$. Set

$$H(z) = \begin{cases} \sum_{k=1}^N u(f_k(z)), & z \in R, \\ 0, & z \in \partial R. \end{cases}$$

$H$ is a continuous function on $R \cup \partial R$ which is positive and $C^\infty$ on $R$. Let $E$ be the critical set of $H$ on $R$ and let $\{\varepsilon_n\}$ be a sequence of positive numbers decreasing to zero with no $\varepsilon_n$ contained in $H(E)$. Such numbers exist by Sard’s theorem. Define $R_n$ to be that component of the set $H^{-1}(\varepsilon_n, \infty)$ which contains the point $z_0$. Then $R_1 \subset R_2 \subset \cdots$, the union of all the $R_n$ is $R$, and $f$ maps $R_n$ onto itself for each $n$ since $H \circ f = H$. Further, by the inverse function theorem, $\partial R_n$ is the union of a finite number of disjoint, smooth, simple closed curves. Finally, the genus of each $R_n$ is no more than $g$ since the genus of a surface is the maximal number of simple closed curves that may be deleted from the surface without disconnecting it. We shall show that $f$ has no more than $2 + 2g$ fixed points in any $R_n$; this will complete the proof of the theorem.

Fix $n$ and let $S$ denote $R_n$. Let $S'$ denote the surface obtained by identifying the orbits $\{f_k(z): k = 1, 2, \ldots, N\}$ to a point and let $F$ be the quotient map from $S$ onto $S'$. We may now apply the Riemann-Hurwitz relation [1, Theorem I.2.7] to $F$, $S$, and $S'$:

$$2g = 2d(g' - 1) + 2 + B$$

where $g$ is the genus of $S$, $g'$ the genus of $S'$, $d$ the degree of $F$, and $B$ the total branching number of $F$. In this case, we know that $d = N$ and $B = (N - 1)\alpha$ where $\alpha$ is the number of fixed points of $f$. This yields

$$(N - 1)\alpha = 2g - 2N(g' - 1) - 2 \leq 2g + 2N - 2.$$  

Consequently,

$$\alpha \leq 2 + 2g/(N - 1) \leq 2 + 2g.$$  

This completes the proof of the theorem.
EXAMPLES. 1. The function \( g(z) = z e^z \) maps the surface \( R = \mathbb{C}\setminus\{0\} \) into itself and has infinitely many fixed points.

2. For \( N \geq 2 \), the polynomial \( p(z) = z^N + z - 1 \) maps the sphere onto itself and the plane onto itself and has \( N \) distinct fixed points.

3. Let \( R \) be the compact Riemann surface of genus \( g \) obtained by taking the connected sum of \( g \) tori. The mapping \( f \) obtained by rotating \( R \) by 180° is the hyperelliptic involution of \( R \) and it has \( 2g + 2 \) fixed points. Indeed, the proof of the theorem shows that the number of fixed points on a compact surface can equal \( 2g + 2 \) only when \( N = 2 \); in this case, \( R \) must be hyperelliptic and \( f \) the unique involution of \( R \); see [1, Proposition III.7.9].

REMARKS. 1. The problem of determining the number of fixed points of an analytic function mapping a planar domain into itself was originally brought to our attention by L. A. Rubel.

2. The theorem has a more direct, but deeper proof. Once Theorem 1 is established, we could appeal to a result of B. Maskit [3] which asserts that there is a closed Riemann surface \( R^* \) of genus \( g \) and a conformal embedding of \( R \) into \( R^* \) so that, under this embedding, every conformal self-map of \( R \) is the restriction of a conformal self-map of \( R^* \). An application of the Riemann-Hurwitz theorem then completes the proof, as it did ours.

3. A version of our Theorem 1 is also to be found in [2] as Theorem 4 although the proof is considerably different.

4. The final paragraph of the proof of the Theorem can be modified to prove the following more general result.

PROPOSITION. Let \( f \) be a diffeomorphism of a compact connected surface \( S \), which may have boundary and which has genus \( g \). Suppose that \( f_N \), the \( N \)th iterate of \( f \), is the identity, and \( f_k \) is not the identity if \( 0 < k < N \). If the number \( \alpha \) of fixed points of \( f \) is finite, then \( \alpha \) satisfies \( \alpha \leq 2 + 2g/(N - 1) \).

PROOF. By averaging a metric we may assume \( f \) is an isometry. Let \( S' \) denote the surface obtained by identifying orbits of \( f \) to a point and let \( F : S \rightarrow S' \) be the quotient map. Since \( f \) is an isometry, in a neighborhood of a fixed point it is a rotation through an angle of \( 2\pi M/N \). (If it was a reflection there would be infinitely many fixed points.) The integer \( M \) must be prime to \( N \) since otherwise for some \( k, 0 < k < N \), \( f_k \) is the identity on a neighborhood of the fixed point and hence on all of \( S \). The remainder of the proof now is the same as that of the Theorem. We note that the Riemann-Hurwitz relation holds in this more general context, and the proof in [1] is adequate to show this.

REFERENCES


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