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POWERS OF GENERATORS OF HOLOMORPHIC SEMIGROUPS

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ABSTRACT. We show that when the (possibly unbounded) linear operator $-A$ generates a bounded holomorphic semigroup of angle θ, and $n(\pi/2 - \theta) < \pi/2$, then $-A^n$ generates a bounded holomorphic semigroup of angle $\pi/2 - n(\pi/2 - \theta)$. When $-A$ generates a bounded holomorphic semigroup of angle $\pi/2$, then, for all n, $-A^n$ generates a bounded holomorphic semigroup of angle $\pi/2$.

Introduction. A strongly continuous semigroup of operators is *holomorphic* if it extends to a holomorphic semigroup in a sector of the form $S_\psi \equiv \{z \,|\, |\arg(z)| < |\psi|\}$ (see Definition 1). Reference [**3**, Chapter 5] has numerous characterizations of holomorphic semigroups.

In this paper, we show that when $-A$ generates a bounded holomorphic semigroup of angle θ (see Definition 1), and $n(\pi/2 - \theta) < \pi/2$, then $-A^n$ generates a bounded holomorphic semigroup of angle $\pi/2 - n(\pi/2 - \theta)$. An immediate corollary is that when $-A$ generates a bounded holomorphic semigroup of angle $\pi/2$, then all powers of A also generate such a semigroup.

Our results are closely related to results in [**1**], where Goldstein proved that if $-A$ generates a holomorphic semigroup of angle $\alpha > \pi/4$, then $-A^2$ generates a holomorphic semigroup of angle $2\alpha - \pi/2$ (see Theorem 4 of our paper, letting $n = 2$). Goldstein also shows that if the Cauchy problem for $u'' + Au = 0$ is well posed, so that $-A$ generates a holomorphic semigroup of angle $\pi/2$, then the same is true of $-A^{2n}$, for $n = 1, 2, \ldots$ (see Corollary 5, for even n).

We will use a generalization of the Dunford functional calculus to construct the semigroup generated by $-A^n$ (see $(*)$ in the proof of Theorem 4). Since our construction is almost identical to [**2**, pp. 249–252], we will refer to that reference extensively to avoid repeating what is written there.

General facts about accretive operators and generators of semigroups may be found in [**2**, §10.8], or [**4**, Chapter 9].

We conclude with an open question.

The referee has pointed out the following application. By taking A to be a second order elliptic operator, our results explain why the Cauchy problem for $u' + A^n u = 0$ is well posed, where A^n is an elliptic operator of order $2n$. We are indebted to the referee for this, and for bringing reference [**1**] to our attention.

All operators are linear, on a Banach space X. The vector space $D(A)$ is the domain of the operator A.

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DEFINITION 1. The set $S_\psi \equiv \{z \mid |\arg(z)| < |\psi|\}$ ("arg" ≡ "argument").

DEFINITION 2. A strongly continuous bounded semigroup $\{T(t)\}_{t \geq 0}$ is a *bounded holomorphic semigroup of angle* θ if it extends to a semigroup $T(z)$, analytic in S_θ, such that, for any positive $\psi < \theta$, $\{\|T(z)\| \mid z \text{ is in } S_\psi\}$ is bounded, and $T(z)x \to x$, as $z \to 0$ in S_ψ, for all x in X.

To show that $-A$ generates a bounded holomorphic semigroup of angle θ, it is sufficient to show that, whenever $|\psi| < \theta$, $-e^{i\psi}A$ generates a strongly continuous bounded semigroup, and $\{\|e^{-zA}\| \mid z \text{ is in } S_\psi\}$ is bounded. (See [**2**, Theorem X.52], and the discussion preceding it.)

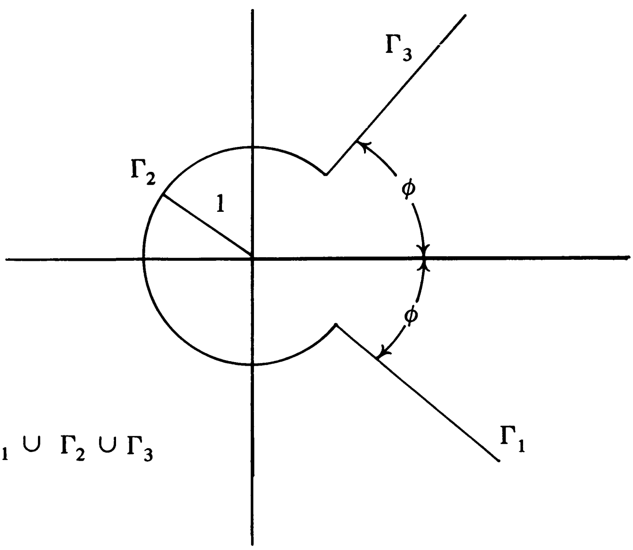
LEMMA 3. *Suppose* $1 + e^{i\psi}x^n = 0$, *where* $|\psi| < \pi/2 - n(\pi/2 - \theta)$. *Then* x *is not in* $S_{(\pi/2-\theta)}$.

PROOF. We have that $x = e^{2\pi ij/n}e^{i(\pi-\psi)/n}$, for some integer j between 0 and $(n-1)$. It is sufficient to show that $(\pi - \psi)/n > \pi/2 - \theta$, and $(\pi - \psi)/n - 2\pi/n < -(\pi/2 - \theta)$, since $(\pi - \psi)/n \leq 2\pi j/n + (\pi - \psi)/n \leq 2\pi - 2\pi/n + (\pi - \psi)/n$, for $0 \leq j \leq (n-1)$. This follows by using the inequality in the hypothesis, to obtain

$$\frac{\pi}{2n} + \left(\frac{\pi}{2} - \theta\right) < \frac{1}{n}(\pi - \psi) < \frac{3\pi}{2n} - \left(\frac{\pi}{2} - \theta\right).$$

THEOREM 4. *Suppose* $-A$ *generates a bounded holomorphic semigroup of angle* θ, *and* $n(\pi/2 - \theta) < \pi/2$. *Then* $-A^n$ *generates a bounded holomorphic semigroup of angle* $\pi/2 - n(\pi/2 - \theta)$.

PROOF. It is sufficient to show that $-e^{i\psi}A^n$ generates a strongly continuous bounded semigroup whenever $|\psi| < \pi/2 - n(\pi/2 - \theta)$, and that $\{\|e^{-zA^n}\| \mid z \text{ is in } S_\psi\}$ is bounded (see comments after Definition 2). Fix ψ, with $|\psi| < \pi/2 - n(\pi/2 - \theta)$. Choose $\phi > 0$ such that $(|\psi| + n\phi) < \pi/2$, $\phi > \pi/2 - \theta$. (The conditions on ψ guarantee that such a ϕ exists.) As in [**2**, Figure 10.5], let Γ be the path below.



$\Gamma \equiv \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$

For z in $\overline{S_\psi}$, define the operator $T(z)$ by

$$(*) \qquad T(z) \equiv \frac{-1}{2\pi i}\int_\Gamma e^{-zw^n}(w-A)^{-1}\,dw.$$

We will show the following:

(1) $\{\|T(z)\| \mid z \text{ is in } \overline{S_\psi}\}$ is bounded.

(2) $\{T(re^{i\psi})\}_{r\geqslant 0}$ is a strongly continuous semigroup.

(3) For all x in $D(A^n)$, $\lim_{r\downarrow 0}(x - T(re^{i\psi})x)/r = e^{i\psi}A^n x$.

Assertion (1) follows as in [**2**, pp. 249–250], using the fact that, for z in $\overline{S_\psi}$, w in $\Gamma_1 \cup \Gamma_3$,

$$\operatorname{Re}(zw^n) = \operatorname{Re}(z|w|^n e^{\pm in\phi}) = |zw^n|(\cos(\arg(z) \pm n\phi)) \geqslant |zw^n|\cos(|\psi| + n\phi).$$

Assertion (2) follows exactly as in [**2**, p. 251]. To prove assertion (3), first suppose $r > 0$, and x is in $D(A^n)$. Then

$$(-2\pi i)\frac{d}{dr}T(re^{i\psi})x = \int_\Gamma \frac{d}{dr}\left(e^{-re^{i\psi}w^n}\right)(w-A)^{-1}x\,dw$$

(since $e^{-re^{i\psi}w^n}$ decays exponentially)

$$= \int_\Gamma e^{-re^{i\psi}w^n}(-e^{i\psi}w^n)(w-A)^{-1}x\,dw$$

$$= \int_\Gamma e^{-re^{i\psi}w^n}(-e^{i\psi}A^n)(w-A)^{-1}x\,dw - e^{i\psi}\sum_{k=1}^{n}\int_\Gamma e^{-re^{i\psi}w^n}w^{n-k}A^{k-1}x\,dw$$

(since $w^n - A^n = (w-A)\sum_{k=1}^n w^{n-k}A^{k-1}$)

$$= \int_\Gamma e^{-re^{i\psi}w^n}(w-A)^{-1}(-e^{i\psi}A^n x)\,dw$$

(by Cauchy's theorem)

$$= (2\pi i)T(re^{i\psi})(e^{i\psi}A^n x).$$

Assertion (3) now follows as in [**2**, p. 252], by writing $x - T(re^{i\psi})x = -\int_0^r d/ds\,T(se^{i\psi})x\,ds$. It now follows that some extension of $e^{i\psi}A^n$ generates the strongly continuous bounded semigroup $\{T(re^{i\psi})\}_{r\geqslant 0}$. Thus $e^{i\psi}A^n$ is accretive, with respect to the equivalent norm $|||x||| \equiv \sup_{r\geqslant 0}\|T(re^{i\psi})x\|$. To show that $e^{i\psi}A^n$ generates $T(re^{i\psi})$, it is sufficent to show that $(I + e^{i\psi}A^n)$ is onto X (see [**4**, Chapter 9.8]—note that an operator B is *dissipative* if and only if $(-B)$ is accretive). By the lemma, when we factor

$$(1 + e^{i\psi}x^n) = \prod_{j=1}^{n}(x - \alpha_j),$$

we have $(A - \alpha_j I)$ invertible, for all j. Thus, since $(A - \alpha_j I)$ takes $D(A^k)$ onto $D(A^{k-1})$, for all k, j, we have that $(I + e^{i\psi}A^n) = \prod_{j=1}^n(A - \alpha_j I)$ takes $D(A^n)$ onto X, as desired. Finally, $\{\|e^{-zA^n}\| \mid z \text{ is in } S_\psi\}$ is bounded, by assertion (1), proving the theorem.

COROLLARY 5. *Suppose $-A$ generates a bounded holomorphic semigroup of angle $\pi/2$. Then, for any natural number n, $-A^n$ generates a bounded holomorphic semigroup of angle $\pi/2$.*

REMARK. We could define fractional powers of A, as the generator of the semigroup defined by $(*)$, in the proof of Theorem 4, with n equal to a nonintegral positive number. When $0 < r < 1$, and $-A$ generates a strongly continuous bounded (not necessarily holomorphic) semigroup, the same proof shows that the fractional power $-A^r$ thus defined generates a bounded holomorphic semigroup of angle $(1 - r)\pi/2$. Other formulas for fractional powers appear in [**4**, Chapter 9.11].

OPEN QUESTION. Suppose that, for all n, $-A^n$ generates a strongly continuous bounded (not necessarily holomorphic) semigroup. Does it follow that $-A$ generates a bounded holomorphic semigroup of angle $\pi/2$?

Arguing as in the "remark" above, we could show that, for all n, there exists an operator B_n that generates a bounded holomorphic semigroup of angle $(1 - 1/n)\pi/2$ such that $(B_n)^n = A^n$. If we could deduce that $-A$ also generates a bounded holomorphic semigroup of angle $(1 - 1/n)\pi/2$ for all n, this would answer the "open question" in the affirmative. It would be sufficient to have the converse of Theorem 4 (including the case when $n(\pi/2 - \theta) = \pi/2$—define a holomorphic semigroup of angle 0 to be a strongly continuous semigroup) be true.

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