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ON PERIODIC SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS WITH SINGULARITIES

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ABSTRACT. Necessary and sufficient conditions for existence of periodic solutions of differential equations containing singularities are given. Our theorems apply to $u'' + 1/u^\alpha = h(t) \equiv h(t+T)$ for all $\alpha > 0$ and to $u'' - 1/u^\alpha = h(t)$ if $\alpha \geq 1$, and for this case $\alpha \geq 1$ is an essential condition.

1. In this note we consider a class of second order scalar differential equations with periodic forcing, zero damping, and a restoring force which becomes infinite at a finite displacement, which we take to be zero. We give a necessary and sufficient condition for the existence of a periodic solution for equations in this class. Our first theorem will show that if $h(t)$ is continuous and T-periodic, then for all $\alpha > 0$ there exists a positive T-periodic solution of

$$u''(t) + 1/u(t)^\alpha = h(t) \tag{1.1}$$

if and only if $h(t)$ has a positive mean value. Our second theorem will show that if $\alpha \geq 1$, then the repulsive type equation

$$u''(t) - 1/u(t)^\alpha = h(t) \tag{1.2}$$

has a positive T-periodic solution if and only if $h(t)$ has a negative mean value.

In the last section, we show that this result is the best possible, by showing that for any α, $0 < \alpha < 1$, we can choose h so that h has negative mean value and the equation has no T-periodic solution.

Our methods consist of sub- and super-solution arguments and truncation arguments based on a priori upper and lower bounds of periodic solutions which permit reduction to the case of bounded nonlinearities and the application of the results in [**3**] (see also [**1**, p. 121 or **4**, p. 23]).

2. In this section we consider a general class of problems which includes (1.1) with $\alpha > 0$.

THEOREM 2.1. *Let g be a real valued continuous function defined on $(-\infty, 0) \cup (0, \infty)$ such that $g(\xi) \to 0$ as $|\xi| \to \infty$, $g(\xi) \to +\infty$ as $\xi \to 0+$, $g(\xi) \to -\infty$ as $\xi \to 0-$, and $g(\xi)\xi > 0$ for $\xi \neq 0$. Let $h(t)$ be defined and continuous for $-\infty < t < \infty$ and satisfy $h(t) \equiv h(t+T)$ for some $T > 0$. A necessary and sufficient condition that there exists a T-periodic solution of*

$$u^{*\prime\prime}(t) + g(u(t)) = h(t) \tag{2.2}$$

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is that

$$\int_0^T h(s)\,ds \neq 0 \tag{2.3}$$

Before proving the theorem let us make clear that by a solution of (2.2) we mean a C^2-function which satisfies the differential equation.

PROOF. Suppose $u(t)$ is a T-periodic solution of (2.2). Since $u(t) \neq 0$ for all t the assumptions on g imply that $g(u(t))$ is either always positive or always negative. Integrating both sides of (2.2) from $t = 0$ to $t = T$ we obtain

$$\int_0^T h(t)\,dt = \int_0^T g(u(t))\,dt \neq 0. \tag{2.4}$$

Therefore (2.3) is necessary for the existence of a T-periodic solution.

Conversely, suppose that (2.3) holds. We shall consider only the case

$$\frac{1}{T}\int_0^T h(t)\,dt \equiv h_0 > 0 \tag{2.5}$$

and show that this implies the existence of a positive T-periodic solution of (2.2). The proof that $h_0 < 0$ implies the existence of a negative T-periodic solution of (2.2) is similar.

If $\varepsilon > 0$ is chosen so small that $g(\varepsilon) - h(t) > 0$ for all t, then the constant function $u_*(t) \equiv \varepsilon$ is a *sub-solution* of the boundary value problem given by equation (2.2) and T-periodic boundary conditions since

$$u^{*\prime\prime}(t) + g(u_*(t)) \geq h(t).$$

(See, for example, [**2**] for a discussion of the method of sub- and super-solutions applied to problems with periodic boundary conditions.) To prove the existence of a positive periodic solution of (2.2) it is only necessary to find a T-periodic C^2 function $u^*(t)$ such that

$$u^{*\prime\prime}(t) + g(u^*(t)) \leq h(t) \tag{2.6}$$

and $u_*(t) < u^*(t)$ for all T. ($u^*(t)$ is a *super-solution.*)

Since the continuous T-periodic function $h(t) - h_0$ has mean value zero, there exists a C^2-function $w(t)$ which is T-periodic such that $w''(t) = h(t) - h_0$. Using the fact that $g(\xi) \to 0$ as $\xi \to +\infty$ we may choose a constant $c > 0$ so large that $u^*(t) \equiv c + w(t) > u_*(t)$ for all t and $g(u^*(t)) < h_0$ for all t. It then follows that (2.6) holds and by earlier remarks this proves the existence of a T-periodic solution $u(t)$ of (2.2).

3. In this section we consider a class of problems which includes (1.2) if $\alpha > 1$. Here we use a truncation argument which also applies to equations of the type (1.1), provided $\alpha > 1$, and which gives the existence of a T-periodic C^1-function u which is strictly positive everywhere and which solves the equation in a weak sense for every $h \in L^1(0,T)$. Analogously one can get negative solutions of (1.2). We use $\|\cdot\|_p$ to indicate the L^p norm on $[0,T]$.

PROPOSITION 3.1. *Let u be a T-periodic distribution and let $u'' \in L^1(0,T)$. Then if one indicates by $(u'')^+$ (resp. $(u'')^-$) the positive (resp. negative) part of u'', then for both the $+$ and the $-$ sign one has*

$$\|u'\|_\infty < \|(u'')^\pm\|_1. \tag{3.2}$$

PROOF. Let t_0 be a point of minimum for u. Therefore, since $u \in C^1$, one has

$$u'(t_0) = u'(t_0 + T) = 0. \tag{3.3}$$

We fix $t \in [t_0, t_0 + T]$ and obtain

$$u'(t) = \int_{t_0}^t u''(s)\,ds \le \int_{t_0}^t (u''(s))^+\,ds \le \int_{t_0}^{t_0+T} (u''(s))^+\,ds = \|(u'')^+\|_1, \tag{3.4}$$

and analogously

$$\begin{aligned} u'(t) &= \int_{t_0+T}^t (u''(s))\,ds = -\int_t^{t_0+T} (u''(s))\,ds \ge -\int_t^{t_0+T} (u''(s))^+\,ds \\ &\ge -\int_{t_0}^{t_0+T} (u''(s))^+\,ds = -\|(u'')^+\|_1. \end{aligned} \tag{3.5}$$

(3.4) and (3.5) give the estimate (3.2) for the $+$ sign. Interchanging u with $-u$, one proves (3.2) with the $-$ sign. □

The preceding result gives an a priori bound from below for any classical positive T-periodic solution of

$$u'' - g(u) = h(t), \tag{3.6}$$

where g is a positive function defined on $(0, +\infty)$ such that

$$\lim_{s \to 0^+} g(s) = +\infty, \qquad \int_0^1 g(x)\,dx = +\infty. \tag{g_1}$$

LEMMA 3.7. *Let (g_1) hold. Then for any constant $M > 0$ there exists a constant $\varepsilon > 0$ such that for any T-periodic continuous function h such that $\|h\|_1 < M$ and any T-periodic positive classical solution u of (3.6) one has*

$$\forall t \in \mathbf{R} : u(t) > \varepsilon. \tag{3.8}$$

PROOF. Let $\xi \in \mathbf{R}_+$ be such that

$$\forall x \le \xi : g(x) > T^{-1}M. \tag{3.9}$$

If one integrates both sides of (3.6) one has

$$\int_0^T g(u(t))\,dt = \int_0^T -h(t)\,dt \le M, \tag{3.10}$$

and therefore by (3.9) one sees that there exists $t_1 \in \mathbf{R}$ such that $u(t_1) > \xi$. Now fix $\varepsilon > 0$ in such a way that

$$\int_\varepsilon^\xi g(x)\,dx > 2M^2. \tag{3.11}$$

Multiplying both sides of (3.6) by u' and integrating between t_1 and t we get

$$\int_{t_1}^{t} u''(s)u'(s)\,ds - \int_{t_1}^{t} g(u(s))u'(s)\,ds = \int_{t_1}^{t} h(s)u'(s)\,ds.$$

Since $\|u'\|_\infty \leq \|(g(u)+h)^-\|_1 \leq \|h^-\|_1 \leq M$ because $g(u) \geq 0$ and from (3.2), we see that for $t \in [t_1, t_1 + T_0]$

$$\frac{1}{2}(u'(t))^2 - \frac{1}{2}(u'(t_1))^2 - \int_{u(t_1)}^{u(t)} g(x)\,dx \leq M^2$$

and finally

$$\int_{u(t)}^{u(t_1)} g(x)\,dx \leq M^2 + \frac{(u'(t_1))^2}{2} \leq 2M^2.$$

Since $u(t_1) > \xi$ and g is positive, from (3.11) one gets $u(t) > \varepsilon$. □

REMARK. The assumption (g_1) has been used in the previous lemma only in order to determine ξ and ε in such a way that (3.9) and (3.11) hold. Therefore the estimate is verified only provided (3.9) and (3.11) are true, and g could be defined in all $\mathbf{R}$. Moreover only (3.9) is affected by the values taken by g at the left side of ε. This last observation is the point on which the truncation used in the following theorem is based.

THEOREM 3.12. *Let $h \in L^1(0,T)$ be given and assume h to be T-periodic in $\mathbf{R}$. Suppose that (g_1) holds and that $g > 0$ and*

$$\lim_{x \to +\infty} g(x) = 0. \tag{3.13}$$

Then (3.6) has a T-periodic weak solution iff $\int_0^T h(t)\,dt < 0$.

PROOF. The necessity comes immediately from (3.10) since g is positive. For the sufficiency, first assume h continuous and let $M > \|h\|_1$. We fix ξ, ε in such a way that (3.9) and (3.11) hold. Then put

$$\overline{g}(s) = \begin{cases} g(s) & \text{if } s \geq \varepsilon, \\ g(\varepsilon) & \text{if } \varepsilon \leq s, \end{cases}$$

defining in this way $\overline{g}$ on $\mathbf{R}$. Since (3.9) is of course preserved if we change g with $\overline{g}$ then by Lemma 3.7 and the subsequent remark we know that the solutions of

$$u'' - \overline{g}(u) = h \tag{3.14}$$

are bounded below by ε and therefore they are precisely the positve solutions of (3.6). The resonance theorem proved in [**3**] states that (3.14) has a T-periodic solution provided

$$-g(\varepsilon) = \lim_{x \to -\infty} -\overline{g}(x) < \frac{1}{T}\int_0^T h < \lim_{x \to +\infty} -\overline{g}(x) = 0$$

and this last condition is verified by the assumptions which we are making and by (3.9). Finally if h is not continuous let $(h_n)_n$ be a sequence of continuous functions which converges to h in $L^1(0,T)$, and let $M \geq \|h_n\|_1\ \forall n \in \mathbf{N}$. Then find ξ and ε according to (3.9)–(3.11) and define the truncation $\overline{g}$. For any n the first part of the theorem provides a solution u_n of $u'' - \overline{g}(u) = h_n$, such that $u_n > \varepsilon$. By standard arguments one can pass to the limit and get a solution of (3.6). $\square$

Let us remark that Proposition 3.1 is also useful in order to solve other kinds of periodic equations with positive nonlinearities. If for instance f is positive and $\lim_{s\to+\infty} f(x) = 0$, $\lim_{s\to-\infty} f(s) = +\infty$ (e.g. $f(s) = e^{-s}$), given any $\xi_1 \in \mathbf{R}$ such that one has $f(\xi) > T^{-1}\|h\|_1$ for a given $h \in L^1(0,T)$ and $\xi < \xi_1$, then any positive solution of

$$u'' - f(u) = h \tag{3.15}$$

verifies the estimate $\xi_1 - T\|h\|_1 \leq u(t)$ for all t in $\mathbf{R}$.

The same estimate holds if f is replaced by $\overline{f}$, where

$$\overline{f}(s) = \begin{cases} f(\xi_1 - T\|h\|_1) & \text{if } s \leq \xi_1 - T\|h\|_1, \\ f(s) & \text{if } \xi_1 - T\|h\|_1 < s. \end{cases}$$

Replacing f by $\overline{f}$, by the results in [**3**], one can solve (3.15) if

$$\lim_{s\to-\infty} -\overline{f}(s) = -f(\xi_1 - T\|h\|_1) < \frac{1}{T}\int_0^T h < \lim_{s\to+\infty} -\overline{f}(s) = 0.$$

This last condition is verified provided $\int_0^T h < 0$. Of course this condition is necessary if f is strictly positive everywhere.

4. In this last section we show that the result in the previous theorem cannot be extended to (1.2) for $0 < \alpha < 1$. More generally we assume that g is strictly positive and that

$$\lim_{s\to 0^+} g(s) = +\infty, \qquad \lim_{s\to+\infty} g(s) = 0, \qquad \int_0^1 g(s)\,ds < +\infty. \tag{g_2}$$

If g verifies (g_2) we have

THEOREM 4.1. *$\forall T > 0$ $\exists M_0 > 0$ such that $\forall M > M_0$ $\exists h$, a continuous T-periodic negative function, such that* (3.6) *has no solution and* $-\int_0^T h = M$.

PROOF. For simplicity we let h be a step function; a small regularization of h does not affect the computations below. We take $h = -\varepsilon^{-1}M\chi_{[t_1,t_1+\varepsilon]}$, where ε is a small positive real number and $\chi_{[t_1,t_1+\varepsilon]}$ denotes the characteristic function of the interval $[t_1, t_1+\varepsilon]$ for $t_1 \in \mathbf{R}$. Suppose u solves (3.6) and fix $\xi \in \mathbf{R}$ such that (3.9) holds. As in Lemma 3.7 we see, using (3.10), that $\max u > \xi$. Of course ξ depends only on M. By the result in Proposition 3.1, $|u'|$ is bounded by M. Since $h = 0$ in $[0,T]\backslash[t_1, t_1+\varepsilon]$, we have that in this interval $u'' = g(u) + h = g(u) > 0$. Thus, the point of $\max u$ must belong to $[t_1, t_1+\varepsilon]$. Collecting all this information we have

$$\inf_{[t_1,t_1+\varepsilon]} u \geq \xi - \varepsilon M > \xi/2,$$

provided we choose $\varepsilon < M^{-1}\xi/2$. We also choose ε so small that $\varepsilon \max_{s\geq\xi/2} g(s) < M/2$. We have

$$u'(t_1+\varepsilon) - u'(t_1) = \int_{t_1}^{t_1+\varepsilon} u''(s)\,ds = \int_{t_1}^{t_1+\varepsilon} h(s)\,ds + \int_{t_1}^{t_1+\varepsilon} g(s)\,ds < -\frac{M}{2}. \tag{4.2}$$

Therefore $|u'(t)| > M/4$ for $t = t_1$ or for $t = t_1 + \varepsilon$. Assume that $|u'(t_1 + \varepsilon)| > M/4$ and for simplicity of notation let $t_1 = 0$.

We also observe that if we fix ξ' such that $\sup_{s>\xi'} g(s) < T^{-1}M$, by Proposition 3.1 and by (3.10) we have $\sup u < \xi' + TM$, and ξ' also does not depend on $M \geq \overline{M}$. Let t_0 be a point of $\min u$ on $[0, T]$. In $[0, \varepsilon]$

$$u'' = g(u) + h < \varepsilon^{-1}M/2 - \varepsilon^{-1}M < 0,$$

so $t_0 \in [\varepsilon, T]$, and therefore $h = 0$ in $[\varepsilon, t_0]$. Multiplying (3.6) by u' and integrating between ε and t_0 we get

$$u'(\varepsilon)^2 - u'(t_0)^2 = 2\int_{u(t_0)}^{u(\varepsilon)} g(s)\,ds \leq 2\int_0^{\xi' + TM} g(s)\,ds,$$

which implies

$$\int_0^{\xi' + TM} g(s)\,ds \geq \frac{M^2}{32}. \tag{4.3}$$

The assumption (g_2) clearly implies that the left-hand side of (4.3) is a sublinear function of M, so (4.3) is definitely false for M large and the theorem is proved.

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