THE ANALYTIC RADON-NIKODYM PROPERTY IN LEBESGUE BOCHNER FUNCTION SPACES

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ABSTRACT. Let \( X \) be a complex Banach space, \( (\Omega, \Sigma, \mu) \) a finite measure space, and \( 1 \leq p < \infty \). Then \( L_p(\mu; X) \) has the analytic Radon-Nikodym property if and only if \( X \) has it.

1. Introduction. In [2] Bukhvalov and Danilevich introduced the notion of the analytic Radon-Nikodym property. The relationship between the analytic Radon-Nikodym property and the existence of bounded holomorphic embeddings has been studied by Aurich [1], while its relationship with representable operators has been studied by Dowling [5]. Recently, Edgar [7] has shown that a complex Banach space has the analytic Radon-Nikodym property if and only if every \( L_1 \)-bounded analytic martingale in \( X \) converges. In this note we are going to show that if a complex Banach space \( X \) has the analytic Radon-Nikodym property then so does \( L_p(\mu; X) \) for \( 1 \leq p < \infty \), where \( (\Omega, \Sigma, \mu) \) is a finite measure space.

2. Preliminaries.

DEFINITION 1. A complex Banach space \( X \) has the analytic Radon-Nikodym property if every \( X \)-valued measure of bounded variation, defined on the Borel subsets of \( T (= \{ z \in \mathbb{C}: |z| = 1 \}) \), whose negative Fourier coefficients vanish, has a Radon-Nikodym derivative with respect to normalized Lebesgue measure \( d\theta/2\pi \) on \( T \).

DEFINITION 2. Let \( X \) be a complex Banach space. The space \( H^p(X) \) consists of all holomorphic functions \( f: D \rightarrow X \), where \( D \) is the open unit disc in \( \mathbb{C} \), and satisfying \( \| f \|_p < \infty \) where

\[
\| f \|_p = \sup_{0 \leq r < 1} \left( \int_0^{2\pi} \| f(re^{i\theta}) \|^p \frac{d\theta}{2\pi} \right)^{1/p}, \quad 0 < p < \infty,
\]

\[
\| f \|_\infty = \sup_{z \in D} \| f(z) \|.
\]

For details on vector-valued holomorphic functions see [4, 8]. \( (H^p(X), \| \cdot \|_p) \) is a Banach space for \( 1 \leq p \leq \infty \).

THEOREM 1 [2]. Let \( X \) be a complex Banach space. The following are equivalent:

(a) \( X \) has the analytic Radon-Nikodym property,
(b) for some \( p \), \( 1 \leq p \leq \infty \), every function in \( H^p(X) \) has radial limits a.e.,
(c) for all \( p \), \( 1 \leq p \leq \infty \), every function in \( H^p(X) \) has radial limits a.e.,

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(d) for some \( p, 1 \leq p < \infty \), \( \lim_{r \to 1} f_r \) exists in \( L_p(T; X) \) for every \( f \in H^p(X) \),  where \( f_r(e^{i\theta}) = f(re^{i\theta}) \),  
(e) for all \( p, 1 \leq p < \infty \), \( \lim_{r \to 1} f_r \) exists in \( L_p(T; X) \) for every \( f \in H^p(X) \).

3. The analytic Radon-Nikodym property for \( L_p(\mu; X) \).

**Theorem 2.** Let \( X \) be a complex Banach space, \( 1 \leq p < \infty \), and \((\Omega, \Sigma, \mu)\) a finite measure space. Then \( L_p(\mu; X) \) has the analytic Radon-Nikodym property if \( X \) does.

**Proof.** Let \( f \in H^p(L_p(\mu; X)) \). Then \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), where \( a_n \in L_p(\mu; X) \) for all \( n \in \mathbb{N} \). Since \( f \) is holomorphic we have that, for each \( 0 < r < 1 \),

\[
\sum_{n=0}^{\infty} \left| a_n \right| r^n < \infty,
\]

that is, for each \( 0 < r < 1 \),

\[
\sum_{n=0}^{\infty} \left( \int_{\Omega} \left| a_n(\omega) \right|^p d\mu(\omega) \right)^{1/p} r^n < \infty.
\]

Hence, for each \( 0 < r < 1 \),

\[
\sum_{n=0}^{\infty} \left( \int_{\Omega} \left| a_n(\omega) \right|^p d\mu(\omega) \right) r^n < \infty,
\]

and so, for each \( 0 < r < 1 \),

\[
\int_{\Omega} \left( \sum_{n=0}^{\infty} \left| a_n(\omega) \right|^p r^n \right) d\mu(\omega) < \infty.
\]

Now, for each \( 0 < r < 1 \),

\[
\sum_{n=0}^{\infty} \left| a_n(\omega) \right|^p r^n < \infty,
\]

for almost all \( \omega \in \Omega \), and from this we get that, for each \( 0 < r < 1 \),

\[
\sum_{n=0}^{\infty} \left| a_n(\omega) \right|^r < \infty,
\]

for almost all \( \omega \in \Omega \). Thus, the function \( f(\cdot)(\omega): D \to X \), defined by \( f(z)(\omega) = \sum_{n=0}^{\infty} a_n(\omega) z^n \), is holomorphic for almost all \( \omega \in \Omega \).

\[
\infty > \|f\|_p^p = \sup_{0 \leq r < 1} \int_0^{2\pi} \left\| f(re^{i\theta}) \right\|_p^p d\theta \frac{d\theta}{2\pi} = \lim_{r \to 1} \int_0^{2\pi} \left\| f(re^{i\theta}) \right\|_p^p d\theta \frac{d\theta}{2\pi}
\]

(since \( \|f(\cdot)\| \) is subharmonic)

\[
= \lim_{r \to 1} \int_0^{2\pi} \int_{\Omega} \left\| f(re^{i\theta})(\omega) \right\|_p^p d\mu(\omega) \frac{d\theta}{2\pi}
\]

\[
= \lim_{r \to 1} \int_0^{2\pi} \int_{\Omega} \left\| f(re^{i\theta})(\omega) \right\|_p^p d\mu(\omega) \frac{d\theta}{2\pi}
\]

\[
= \int_{\Omega} \left( \lim_{r \to 1} \int_0^{2\pi} \left\| f(re^{i\theta})(\omega) \right\|_p^p d\theta \right) d\mu(\omega)
\]
(by the Monotone convergence theorem)

\[ \int_{\Omega} \left( \sup_{0 \leq r < 1} \int_0^{2\pi} \| f(re^{i\theta})(\omega) \|_p \frac{d\theta}{2\pi} \right) d\mu(\omega) \]

\[ = \int_{\Omega} \| f(\cdot)(\omega) \|_p^p d\mu(\omega). \]

Therefore, for almost all \( \omega \in \Omega \), \( \| f(\cdot)(\omega) \|_p < \infty \); that is, for almost all \( \omega \in \Omega \), \( f(\cdot)(\omega) \in H^p(X) \). By Theorem 1 and the assumption that \( X \) has the analytic Radon-Nikodym property we have that, for almost all \( \omega \in \Omega \), \( \lim_{r \uparrow 1} f_r(\cdot)(\omega) \) exists in \( L_p(T; X) \). Moreover,

\[ \| f_r - f_s \|^p_{L_p(T; L_p(\mu; X))} = \int_0^{2\pi} \| f(re^{i\theta}) - f(se^{i\theta}) \|^p \frac{d\theta}{2\pi} \]

\[ = \int_0^{2\pi} \int_{\Omega} \| f(re^{i\theta})(\omega) - f(se^{i\theta})(\omega) \|_p d\mu(\omega) \frac{d\theta}{2\pi} \]

\[ = \int_{\Omega} \int_0^{2\pi} \| f(re^{i\theta})(\omega) - f(se^{i\theta})(\omega) \|_p \frac{d\theta}{2\pi} d\mu(\omega) \]

\[ = \int_{\Omega} (\| f_r(\cdot)(\omega) - f_s(\cdot)(\omega) \|_{L_p(T; X)}^p) d\mu(\omega). \]

But

\[ \| f_r(\cdot)(\omega) - f_s(\cdot)(\omega) \|_{L_p(T; X)} \to 0 \]

as \( r, s \uparrow 1 \) for almost all \( \omega \in \Omega \) and all

\[ \| f_r(\cdot)(\omega) - f_s(\cdot)(\omega) \|_{L_p(T; X)} \leq 2 \| f(\cdot)(\omega) \|_p. \]

Hence by the dominated convergence theorem,

\[ \| f_r - f_s \|_{L_p(T; L_p(\mu; X))}^p \to 0 \]

as \( r, s \uparrow 1 \). Thus \( \lim_{r \uparrow 1} f_r \) exists in \( L_p(T; L_p(\mu; X)) \) and so, by Theorem 1, \( L_p(\mu; X) \) has the analytic Radon-Nikodym property.

REMARK. In [3] it is shown that \( L_p(\mu; X) \) has the Radon-Nikodym property if \( X \) has it and if \( 1 < p < \infty \). Theorem 2 is the analogue of this result but the method of proof is quite different.

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REFERENCES


7. _, Analytic martingale convergence, preprint.

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