ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF THE SECOND ORDER DIFFERENCE EQUATION

ANDRZEJ DROZDOWICZ AND JERZY POPENDA

ABSTRACT. The second order difference equation

\[
\Delta^2 x_n + p_n f(x_n) = 0
\]

is considered. The results give a necessary and sufficient condition for some solution of (E) to have asymptotic behavior \( x_n \sim C = \text{const. as } n \to \infty \).

Introduction. The asymptotic behavior of the solutions of second order differential equations have been considered by R. A. Moore and Z. Nehari [4], W. F. Trench [9], and P. Waltman [10]. The next results for nth order nonhomogeneous differential equations was given by T. G. Hallam [1, 2]. Similar problems with regard to second order difference equations were investigated by J. W. Hooker and W. T. Patula [3] and J. Popenda [7].

In this paper the asymptotic behavior of solutions of the second order difference equation

\[
\Delta^2 x_n + p_n f(x_n) = 0
\]

will be considered. A necessary and sufficient condition for some solution \( x \) of (E) to have the asymptotic behavior

\[
\lim_{n \to \infty} x_n = C,
\]

where \( C \) is a constant such that \( f(C) \neq 0 \), will be proved.

1. A necessary condition.

THEOREM 1. A necessary condition for the existence of a solution \( x \) of (E) which possesses asymptotic behavior (AB) is

\[
\sum_{j=1}^{\infty} j p_j < \infty.
\]

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PROOF. Let \( x \) denote a solution of (E) having the property (AB), i.e. \( x_n \to C \) for \( n \to \infty \). Then

\[
\Delta x_n \to 0 \quad \text{as} \quad n \to \infty.
\]

Assume that \( f(C) > 0 \). (The case \( f(C) < 0 \) with some modifications can be considered in a similar way.) The continuity of \( f \) implies that there exists \( \varepsilon > 0 \) such that \( f(t) > 0, t \in I := [C - \varepsilon, C + \varepsilon] \) for some \( \varepsilon > 0 \). Since \( x_n \to C \) as \( n \to \infty \), there exists \( n_1 = N(\varepsilon) \) such that for each \( n \geq n_1 \), \( x_n \in [C - \varepsilon, C + \varepsilon] \). Therefore

\[
f(x_n) \geq C_0 := \min_{t \in I} f(t) > 0 \quad \text{for} \quad n \geq n_1.
\]

Hence

\[
\Delta x_n - \Delta x_k = - \sum_{j=k}^{n-1} p_j f(x_j) \leq - C_0 \sum_{j=k}^{n-1} p_j \quad \text{for} \quad k \geq n_1.
\]

Using (1.1) we get

\[
C_0 \sum_{j=k}^{n} p_j \leq \Delta x_k \quad \text{for} \quad k \geq n_1.
\]

Therefore the series \( \sum_{j=0}^{n} p_j \) is convergent. Summing (1.2) over \( n \) and tending to infinity with an upper limit we yield

\[
C_0 \sum_{j=n_1}^{n} \sum_{i=0}^{n} p_i \leq C - x_n.
\]

From this fact it follows that the series \( \sum_{j=n_1}^{n} \sum_{i=0}^{n} p_i \) converges. Since

\[
\sum_{j=n_1}^{n} \sum_{i=0}^{n} p_i = \sum_{j=n_1}^{n} (j + 1 - n_1)p_j,
\]

the series \( \sum_{j=n_1}^{n} (j + 1 - n_1)p_j \) is also convergent. By observing that

\[
\sum_{j=n_1}^{n} j p_j = \sum_{j=n_1}^{n} (j + 1 - n_1)p_j + (n_1 - 1) \sum_{j=n_1}^{n} p_j
\]

we see that the series \( \sum_{j=n_1}^{n} j p_j \) is convergent. Q.E.D.

REMARK 1. From (1.2) it follows that \( \Delta x_k \geq 0 \) for \( k \geq n_1 \). Therefore the solution \( x_n \) is increasing for \( n \geq n_1 \). We see that \( x_l \leq C \) for \( l \geq n_1 \). This result means that if \( f(C) > 0 \) then the solution of (E) which possesses the asymptotic behavior (AB) monotonically approaches \( C \) from below. If \( f(C) < 0 \), then \( x_n \) must monotonically tend to \( C \) from above.

2. A sufficient condition.

THEOREM 2. For every \( k \in N \) let

\[
(*) \quad i_R + p_k f : R \to R \text{ be a surjection (} i_R \text{ denotes an identity function on } R)\.
\]

A sufficient condition for the existence of a solution \( x \) of (E) which possesses the asymptotic behavior (AB) is (NS).

PROOF. The cases \( C > 0 \) and \( f(C) > 0 \) will be considered. (The other cases, i.e. \( C < 0 \) or \( f(C) < 0 \), with some modifications can be shown in a similar way.)
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Let (NS) hold. Hence

\[ \lim_{n \to \infty} \sum_{j=n}^{\infty} j \rho_j = 0. \]

One can observe that the sequence \( \{\sum_{j=n}^{\infty} j \rho_j\}_{n=1}^{\infty} \) is nonincreasing. Analogous to the proof of Theorem 1 there exists an interval \( I = [C-\varepsilon, C+\varepsilon] \) such that \( f(t) > 0 \), \( t \in I \) for some \( \varepsilon > 0 \). Denoting \( C_1 := \max_{t \in I^-} f(t) \), where \( I^- = [C-\varepsilon, C] \) from (2.1), we obtain

\[ C_1 \sum_{j=n}^{\infty} j \rho_j \leq \varepsilon \quad \text{for all } n \geq N(\varepsilon). \]

Let us set

\[ n_2 = \min \left\{ n \in N : C_1 \sum_{j=n}^{\infty} j \rho_j \leq \varepsilon \right\}. \]

Let \( \ell_{\infty} \) denote the Banach space of bounded sequences \( x = \{h_i\}_{i=1}^{\infty} \) with norm \( \|x\| = \sup_{i \geq 1} |h_i| \). Moreover let us define the set \( T \subset \ell_{\infty} \) in the following way:

\[ x = \{h_i\}_{i=1}^{\infty} \in T \quad \text{if} \quad \begin{cases} h_k = C & \text{for } k = 1, 2, \ldots, n_2 - 1, \\ h_k \in I_k^- & \text{for } k \geq n_2, \end{cases} \]

where

\[ I_k^- := \left[ C - C_1 \sum_{j=k}^{\infty} j \rho_j, C \right], \quad k \geq n_2. \]

It is easy to show that \( T \) is bounded, convex and closed in \( \ell_{\infty} \). We will show that \( T \) is compact. Set \( \text{diam}([a, b]) = b - a; \ a, b \in R \). By (NS) it follows that \( \text{diam} I_n^- \to 0 \) for \( n \to \infty \). Choose any \( \varepsilon_1 > 0 \). If \( \varepsilon_1 \) is such that \( \text{diam} I_n^- < \varepsilon_1 \), then the element \( v = \{C, C, C, \ldots\} \in \ell_{\infty} \) is an \( \varepsilon_1 \)-net. The case \( \text{diam} I_n^- \geq \varepsilon_1 \) will be considered. Let \( n_3 \geq n_2 \) be such that \( \text{diam} I_{n_3}^- \geq \varepsilon_1 \) and \( \text{diam} I_{n_3+1}^- < \varepsilon_1 \). (Everyone can find \( n_3 \) because \( \text{diam} I_n^- \to 0 \) for \( n \to \infty \).) Then it is easy to show that the set of elements of the space \( l_{\infty} \) in the form

\[ v_{s_1, s_2, \ldots, s_{n_3-n_2+1}}^{1, 2, \ldots, n_3-n_2+1} = \{C, \ldots, C, C - s_1 \varepsilon_1, \ldots, C - s_{n_3-n_2+1} \varepsilon_1, C, \ldots\} \]

where

\[ s_i = 0, 1, \ldots, r_i := \text{En} \left[ \frac{\text{diam} I_{n_2+i-1}^-}{\varepsilon_1} \right] + 1, \quad i = 1, 2, \ldots, n_3 - n_2 + 1, \]

to set up an \( \varepsilon_1 \)-net. (En denotes an entire function.) One can observe that

\[ \text{card}\{v_{s_1, s_2, \ldots, s_{n_3-n_2+1}}^{1, 2, \ldots, n_3-n_2+1}\} = \prod_{i=1}^{n_3-n_2+1} (r_i + 1) < \infty. \]

Hence the \( \varepsilon_1 \)-net is finite and by the Hausdorff theorem \( T \) is compact.

Let us define the operator \( A \) on \( T \) in the following way:

\[ Ax = y = \{b_1, b_2, \ldots, b_{n_2-1}, b_{n_2}, \ldots, b_k, \ldots\}, \]
where
\[ b_n = \begin{cases} C & \text{for } n = 1, 2, \ldots, n_2 = 1; \\ C - \sum_{j=n}^{\infty} (j + 1 - n)p_jf(h_j) & \text{for } n \geq n_2. \end{cases} \]

We will show that \( A \) is a function from \( T \) to \( T \). By observing that \( I_k^- \subset I^+ \) it follows that \( 0 < f(h_k) \leq C_1 \) for \( k \geq n_2 \). For \( j \geq k \) one obtains the inequality
\[ 0 < (j + 1 - k)p_jf(h_j) \leq jp_jf(h_j) \leq C_1jp_j. \]

Hence
\[ C \geq C - \sum_{j=k}^{\infty} (j + 1 - k)p_jf(h_j) \geq C - C_1 \sum_{j=k}^{\infty} jp_j. \]

Thus \( b_k \in I_k^- \) for \( k \geq n_2 \). Therefore \( y \in T \).

Next we will show that \( A \) is continuous. Since \( f \) is continuous on \( R \), it is uniformly continuous on \( I^- \). Hence for each \( \varepsilon_2 > 0 \) there exists \( \delta_1 > 0 \) such that the condition \( |t_1 - t_2| < \delta_1 \) implies \( |f(t_1) - f(t_2)| < \varepsilon_2 \). Consider the sequence \( \{x_m\}_{m=1}^{\infty}, \ x_m \in T, \) such that
\[ (2.2) \ \|x_m - x_0\| \to 0; \ \text{i.e., sup}_{n \geq 1} |h_n^m - h_n^0| \to 0, \ \text{as } m \to \infty. \]

From (2.2) it follows that there exists \( n_3 = N(\delta_1) \) such that
\[ \|x_m - x_0\| < \delta_1; \ \text{i.e., sup}_{n \geq 1} |h_n^m - h_n^0| < \delta_1 \ \text{for } m \geq n_3. \]

Hence
\[ \forall_{m \geq n_3} \forall_{i \in N} |h_i^m - h_i^0| < \delta_1. \]

Then for \( m \geq n_3 \)
\[ \|Ax_m - Ax_0\| = \sup_{n \geq 1} |b_n^m - b_n^0| \]
\[ = \sup_{n \geq n_2} \left| \sum_{j=n}^{\infty} (j + 1 - n)p_jf(h_j^m) - \sum_{j=n}^{\infty} (j + 1 - n)p_jf(h_j^0) \right|, \]
where \( b^0 = Ax_0 \) and \( b^m = Ax^m \).

Since the series \( \sum_{j=n}^{\infty} (j + 1 - n)p_jf(h_j^m), \sum_{j=n}^{\infty} (j + 1 - n)p_jf(h_j^0) \) are convergent,
\[ \|Ax_m - Ax_0\| \leq \varepsilon_2 \sum_{j=n_2}^{\infty} (j + 1 - n_2)p_j, \ \text{m} \geq n_3. \]

Hence \( A \) is continuous.

By the Schauder fixed point theorem [8] there exists a solution in \( T \) of the equation \( x = Ax \). Let \( z = \{d_1, d_2, \ldots, d_{n_2-1}, d_{n_2}, \ldots\} \) denote such a solution. Since \( z \in T \), it can be written as follows:
\[ z = \{C, C, \ldots, C, d_{n_2}, d_{n_2+1}, \ldots\} \]
and

$$Az = \left\{ C, C, \ldots, C, C - \sum_{j=n_2}^{\infty} (j + 1 - n_2)p_j f(d_j), \right. $$

$$\left. C - \sum_{j=n_2+1}^{\infty} (j - n_2)p_j f(d_j), \ldots \right\}.$$ 

Therefore

$$d_n = C - \sum_{j=n}^{\infty} (j + 1 - n)p_j f(d_j) \quad \text{for } n \geq n_2. \quad (2.3)$$

Applying the operator $\Delta$ to (2.3) we yield

$$\Delta d_n = \sum_{j=n}^{\infty} p_j f(d_j) \quad \text{for } n \geq n_2. \quad (2.4)$$

Hence $\Delta^2 d_n = -p_n f(d_n)$ holds for $n \geq n_2$. This means that the sequence $\{d_n\}_{n=1}^{\infty}$ fulfills the equation (E) but for $n \geq n_2$ only.

We now prove the existence of the solution $\{x_n\}_{n=1}^{\infty}$ of (E) such that $x_n = d_n$ for $n \geq n_2$.

One can observe that (E) can be rewritten as

$$x_n + p_n f(x_n) = -x_{n+1} + 2x_{n+1}. \quad (2.5)$$

If $n = n_2 - 1$ we get

$$x_{n_2-1} + p_{n_2-1} f(x_{n_2-1}) = -x_{n_2+1} + 2x_{n_2}. \quad (2.6)$$

But we demand for $x_n$ to be equal to $d_n$ for $n \geq n_2$.

From (2.6) we obtain

$$x_{n_2-1} + p_{n_2-1} f(x_{n_2-1}) = -d_{n_2+1} + 2d_{n_2}. \quad (2.7)$$

By (*) it follows that the equation

$$x + p_{n_2-1} f(x) = -d_{n_2+1} + 2d_{n_2} \quad (2.8)$$

possesses solutions. Let us denote one of them by $x_{n_2-1}$. Analogously we can calculate $x_{n_2-2}$, $x_{n_2-3}$, $x_2$, $x_1$ one after the other. Consequently we get the sequence which fulfills (2.4), i.e. which also fulfills (E). Moreover this sequence is identical to $\{d_n\}_{n=1}^{\infty}$ for $n \geq n_2$ and it has the asymptotic behavior (AB) because

$$\lim_{n \to \infty} d_n = C. \quad \text{Q.E.D.} \quad (2.9)$$

**REMARK 2.** One can observe that if $f$ is bounded on $\mathbb{R}$ or fulfills the condition $xf(x) > 0$ for $x \neq 0$ then condition (*) is satisfied. From the proof of Theorem 2 we can deduce that (*) may be weakened as follows:

$$i_R + p_k f : R \to R \quad \text{for } k < n_2, \ k \in N. \quad (2.10)$$

**REMARK 3.** If the assumptions of Theorem 2 hold then analogously an existence of a solution of the equation

$$\Delta^2 x_n + p_{n+k} f(x_{n+k}) = 0, \quad k \geq 1, \quad (E_k)$$

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having the asymptotic behavior (AB) may be proved. In this case the operator $A$
similar to the above but with

$$b_n = C - \sum_{j=n+k}^{\infty} (j + 1 - n - k)p_j f(h_j) \quad \text{for } x = \{h_i\}_{i=1}^{\infty} \in T$$

should be defined.

REMARK 4. If (E) possesses a solution $x$ such that $\lim_{n \to \infty} x_n = C$ then equation
(E) has a solution with $\lim_{n \to \infty} x_n = C_2$, where $C_2 \in (C - \epsilon, C + \epsilon) \subset I$.

REMARK 5. If for some $C$, $f(C) = 0$, then independently of the form of $p$,
equation (E) has a solution with (AB). It has the form $x_n = C$ for each $n \geq 1$.
Conversely, if, for each $n \geq n_2$, $x_n = C$ is the solution of (E) then $p_n f(C) = 0$ for
$n \geq n_2$. Hence $f(C) = 0$ or $p_n = 0$ for each $n \geq n_2$. For the second case ($p_n = 0$)
the condition $\sum_{j=1}^{\infty} j p_j < \infty$ obviously holds.

EXAMPLE. The special case $f(x) = x$ and $k = 1$ will be studied. In this case
the equation (E_k) can be written in the following two equivalent forms:

$$(E_1) \quad \Delta^2 x_n + p_{n+1} x_{n+1} = 0, \quad x_{n+2} - q_n x_{n+1} + x_n = 0,$$

where $q_n = 2 - p_{n+1}$, $n \in N$. If $q_n < 2$, $n \in N$ and $\sum_{j=2}^{\infty} (2 - q_{j-1}) j < \infty$ then
(E_1) possesses a solution which asymptotically approaches any positive constant.

Analogously in the case $k = 2$ one obtains the equation

$$(E_2) \quad x_{n+2} - 2q_n x_{n+1} + q_n x_n = 0,$$

where $q_n = 1/(p_{n+2} + 1)$.

If $0 < q_n < 1$ and $\sum_{j=3}^{\infty} (1/q_{j-1} - 1) j < \infty$ then (E_2) possesses a solution which
asymptotically approaches any positive constant.

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INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF POZNAŃ, UL. PIOTROWO
3A, 60-965 POZNAŃ, POLAND